Problem 13

In each of Problems 11 through 14, solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

$$9y'' + 6y' + 82y = 0, \qquad y(0) = -1, \quad y'(0) = 2$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$9(r^2 e^{rt}) + 6(re^{rt}) + 82(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$9r^2 + 6r + 82 = 0$$

$$r = \frac{-6 \pm \sqrt{36 - 4(9)(82)}}{2(9)} = \frac{-6 \pm \sqrt{-2916}}{18} = \frac{-6 \pm 54i}{18} = -\frac{1}{3} \pm 3i$$

$$r = \left\{-\frac{1}{3} - 3i, -\frac{1}{3} + 3i\right\}$$

Two solutions to the ODE are $y = e^{(-1/3-3i)t}$ and $y = e^{(-1/3+3i)t}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_1 e^{(-1/3-3i)t} + C_2 e^{(-1/3+3i)t} \\
&= C_1 e^{-t/3-3it} + C_2 e^{-t/3+3it} \\
&= C_1 e^{-t/3} e^{-3it} + C_2 e^{-t/3} e^{3it} \\
&= C_1 e^{-t/3}[\cos(-3t) + i\sin(-3t)] + C_2 e^{-t/3}[\cos(3t) + i\sin(3t)] \\
&= C_1 e^{-t/3}[\cos(3t) - i\sin(3t)] + C_2 e^{-t/3}[\cos(3t) + i\sin(3t)] \\
&= C_1 e^{-t/3}\cos 3t - iC_1 e^{-t/3}\sin 3t + C_2 e^{-t/3}\cos 3t + iC_2 e^{-t/3}\sin 3t \\
&= (C_1 + C_2)e^{-t/3}\cos 3t + (-iC_1 + iC_2)e^{-t/3}\sin 3t
\end{aligned}$$

Using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(t) = C_3 e^{-t/3}\cos 3t + C_4 e^{-t/3}\sin 3t.$$

Differentiate it with respect to t.

$$y'(t) = -\frac{C_3}{3}e^{-t/3}\cos 3t - 3C_3 e^{-t/3}\sin 3t - \frac{C_4}{3}e^{-t/3}\sin 3t + 3C_4 e^{-t/3}\cos 3t$$

Apply the initial conditions now to determine C_3 and C_4.

$$\begin{aligned}
y(0) &= C_3 = -1 \\
y'(0) &= -\frac{C_3}{3} + 3C_4 = 2
\end{aligned}$$

Solving this system of equations yields $C_3 = -1$ and $C_4 = 5/9$. Therefore,

$$y(t) = -e^{-t/3}\cos 3t + \frac{5}{9}e^{-t/3}\sin 3t.$$

Take the limit of $y(t)$ as $t \to \infty$.

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{t\to\infty}\left(-e^{-t/3}\cos 3t + \frac{5}{9}e^{-t/3}\sin 3t\right)\\ &= \lim_{t\to\infty}\left(-\cos 3t + \frac{5}{9}\sin 3t\right)e^{-t/3}\\ &= 0\end{aligned}$$

Below is a plot of $y(t)$ versus t.