Problem 20

Problems 20 through 22 indicate other ways of finding the second solution when the characteristic equation has repeated roots.

(a) Consider the equation $y'' + 2ay' + a^2y = 0$. Show that the roots of the characteristic equation are $r_1 = r_2 = -a$, so that one solution of the equation is e^{-at}.

(b) Use Abel's formula [Eq. (23) of Section 3.2] to show that the Wronskian of any two solutions of the given equation is

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = c_1e^{-2at},$$

where c_1 is a constant.

(c) Let $y_1(t) = e^{-at}$ and use the result of part (b) to obtain a differential equation satisfied by a second solution $y_2(t)$. By solving this equation, show that $y_2(t) = te^{-at}$.

Solution

Part (a)

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + 2a(re^{rt}) + a^2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2ar + a^2 = 0$$

$$(r + a)^2 = 0$$

$$r = \{-a\}$$

So one solution to the ODE is $y = e^{-at}$.

Part (b)

Suppose that y_1 and y_2 are two solutions to the ODE. They then satisfy

$$y_1'' + 2ay_1' + a^2y_1 = 0$$
$$y_2'' + 2ay_2' + a^2y_2 = 0.$$

Multiply both sides of the first equation by $-y_2$ and multiply both sides of the second equation by y_1.

$$-y_1''y_2 - 2ay_1'y_2 - a^2y_1y_2 = 0$$
$$y_1y_2'' + 2ay_1y_2' + a^2y_1y_2 = 0$$

Add the respective sides of each equation.

$$y_1y_2'' - y_1''y_2 + 2ay_1y_2' - 2ay_1'y_2 = 0$$

Factor $2a$.

$$y_1y_2'' - y_1''y_2 + 2a(y_1y_2' - y_1'y_2) = 0$$

Note that the Wronskian of y_1 and y_2 is defined as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \quad \Rightarrow \quad W'(y_1, y_2) = \cancel{y_1'y_2'} + y_1y_2'' - y_1''y_2 - \cancel{y_1'y_2'} = y_1y_2'' - y_1''y_2,$$

so the previous equation can be written as

$$W' + 2aW = 0.$$

Solve the ODE.

$$W' = -2aW$$
$$\frac{W'}{W} = -2a$$
$$\frac{d}{dt}\ln|W| = -2a$$

Integrate both sides.

$$\ln|W| = -2at + C$$

Exponentiate both sides.

$$\begin{aligned} |W| &= e^{-2at+C} \\ &= e^{-2at}e^{C} \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$W(t) = \pm e^{C}e^{-2at}$$

Therefore, using c_1 for $\pm e^{C}$,

$$W(t) = c_1e^{-2at}.$$

Part (c)

Let $y_1(t) = e^{-at}$, set $c_1 = 1$, and use the definition of the Wronskian to obtain the differential equation for y_2.

$$W(y_1, y_2) = y_1y_2' - y_1'y_2$$
$$e^{-2at} = e^{-at}y_2' - (-ae^{-at})y_2$$
$$e^{-2at} = e^{-at}y_2' + ae^{-at}y_2$$

Multiply both sides by e^{at}.

$$y_2' + ay_2 = e^{-at}$$

To solve this ODE, multiply both sides by the integrating factor,

$$I = \exp\left(\int^t a\,ds\right) = e^{at},$$

to get

$$e^{at}y_2' + ae^{at}y_2 = 1.$$

The left side can be written as $d/dt(Iy_2)$ by the product rule.

$$\frac{d}{dt}(e^{at}y_2) = 1$$

Integrate both sides with respect to t, setting the new integration constant to zero.

$$e^{at}y_2 = t$$

Therefore,

$$y_2(t) = te^{-at}.$$