Problem 24

In each of Problems 23 through 30, use the method of reduction of order to find a second solution of the given differential equation.

\[ t^2y'' + 2ty' - 2y = 0, \quad t > 0; \quad y_1(t) = t \]

Solution

Because this ODE is homogeneous, any constant multiple of \( y_1(t) \) is also a solution: \( cy_1(t) = ct \). According to the method of reduction of order, the general solution is obtained by allowing \( c \) to vary as a function of \( t \).

\[ y(t) = c(t)t \]

Substitute this formula for \( y(t) \) into the ODE.

\[ t^2[c(t)t]' + 2t[c(t)t]' - 2[c(t)t] = 0 \]

Evaluate the derivatives using the product rule.

\[ t^2[c'(t)t + c(t)]' + 2t[c'(t)t + c(t)] - 2[c(t)t] = 0 \]

\[ t^3c''(t) + 2t^2c'(t) + 2tc(t) - 2tc(t) = 0 \]

Divide both sides by \( t^3 \).

\[ c''(t) + \frac{4}{t}c'(t) = 0 \]

This is a linear first-order ODE for \( c'(t) \), so it can be solved by multiplying both sides by an integrating factor \( I \).

\[ I = \exp \left( \int \frac{4}{s} \, ds \right) = e^{4\ln t} = e^{\ln t^4} = t^4 \]

Proceed with the multiplication.

\[ t^4c''(t) + 4t^3c'(t) = 0 \]

The left side can be written as \( d/dt[Ic'(t)] \) by the product rule.

\[ \frac{d}{dt}[t^4c'(t)] = 0 \]

Integrate both sides with respect to \( t \).

\[ t^4c'(t) = C_1 \]

Divide both sides by \( t^4 \).

\[ c'(t) = \frac{C_1}{t^4} \]

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Integrate both sides with respect to $t$ once more.

$$c(t) = -\frac{C_1}{3t^3} + C_2$$

Therefore, using a new constant $C_3$ for $-C_1/3$, the general solution is

$$y(t) = \frac{C_3}{t^2} + C_2t;$$

the second solution is $y_2(t) = 1/t^2$. 