# Problem 25

In each of Problems 23 through 30, use the method of reduction of order to find a second solution of the given differential equation.

$$t^2y'' + 3ty' + y = 0, \quad t > 0; \qquad y_1(t) = t^{-1}$$

---

### Solution

Because this ODE is homogeneous, any constant multiple of $y_1(t)$ is also a solution: $cy_1(t) = ct^{-1}$. According to the method of reduction of order, the general solution is obtained by allowing $c$ to vary as a function of $t$.

$$y(t) = c(t)t^{-1}$$

Substitute this formula for $y(t)$ into the ODE.

$$t^2[c(t)t^{-1}]'' + 3t[c(t)t^{-1}]' + c(t)t^{-1} = 0$$

Evaluate the derivatives using the product rule.

$$t^2[c'(t)t^{-1} - c(t)t^{-2}]' + 3t[c'(t)t^{-1} - c(t)t^{-2}] + c(t)t^{-1} = 0$$

$$t^2[c''(t)t^{-1} - c'(t)t^{-2} - c'(t)t^{-2} + 2c(t)t^{-3}] + 3t[c'(t)t^{-1} - c(t)t^{-2}] + c(t)t^{-1} = 0$$

$$tc''(t) - 2c'(t) + \cancel{2t^{-1}c(t)} + 3c'(t) - \cancel{3t^{-1}c(t)} + \cancel{t^{-1}c(t)} = 0$$

$$tc''(t) + c'(t) = 0$$

The left side can be written as $d/dt[tc'(t)]$ by the product rule.

$$\frac{d}{dt}[tc'(t)] = 0$$

Integrate both sides with respect to $t$.

$$tc'(t) = C_1$$

Divide both sides by $t$.

$$c'(t) = \frac{C_1}{t}$$

Integrate both sides with respect to $t$.

$$c(t) = C_1 \ln t + C_2$$

Therefore, the general solution is

$$y(t) = C_1t^{-1}\ln t + C_2t^{-1};$$

the second solution is $y_2(t) = t^{-1}\ln t$.