Problem 27

In each of Problems 23 through 30, use the method of reduction of order to find a second solution of the given differential equation.

\[ xy'' - y' + 4x^3y = 0, \quad x > 0; \quad y_1(x) = \sin x^2 \]

Solution

Because this ODE is homogeneous, any constant multiple of \( y_1(x) \) is also a solution: \( cy_1(x) = c\sin x^2 \). According to the method of reduction of order, the general solution is obtained by allowing \( c \) to vary as a function of \( x \).

\[ y(x) = c(x) \sin x^2 \]

Substitute this formula for \( y(x) \) into the ODE.

\[ x[c(x)\sin x^2]'' - [c(x)\sin x^2]' + 4x^3[c(x)\sin x^2] = 0 \]

Evaluate the derivatives using the product rule.

\[ x[c'(x)\sin x^2 + 2xc(x)\cos x^2]' - [c'(x)\sin x^2 + 2xc(x)\cos x^2] + 4x^3[c(x)\sin x^2] = 0 \]

\[ xc''(x)\sin x^2 + 2x^2c'(x)\cos x^2 + 2xc(x)\cos x^2 + 2x^2c'(x)\cos x^2 - 4x^3c(x)\sin x^2 \]

\[ -c'(x)\sin x^2 - 2xc(x)\cos x^2 + 4x^3c(x)\sin x^2 = 0 \]

\[ xc''(x)\sin x^2 + 4x^2c'(x)\cos x^2 - c'(x)\sin x^2 = 0 \]

\[ (x \sin x^2)c''(x) + (4x^2 \cos x^2 - \sin x^2)c'(x) = 0 \]

Solve for \( c''(x)/c'(x) \).

\( (x \sin x^2)c''(x) = (-4x^2 \cos x^2 + \sin x^2)c'(x) \)

\[ \frac{c''(x)}{c'(x)} = -4x \frac{\cos x^2}{\sin x^2} + \frac{1}{x} \]

The left side can be written as \( d/dx[\ln c'(x)] \) by the chain rule.

\[ \frac{d}{dx}[\ln c'(x)] = -4x \frac{\cos x^2}{\sin x^2} + \frac{1}{x} \]

Integrate both sides with respect to \( x \).

\[ \ln c'(x) = \int^x \left(-4s \frac{\cos s^2}{\sin s^2} + \frac{1}{s}\right) ds + C_1 \]

\[ = -\int^x 4s \frac{\cos s^2}{\sin s^2} ds + \int^x \frac{1}{s} ds + C_1 \]

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Make the substitution,
\[ u = \sin^2 s \]
\[ du = 2s \cos s^2 ds, \]
in the first integral and evaluate the second one.

\[
\ln c'(x) = -\int \frac{\sin^2 du}{2u} + \ln x + C_1
\]
\[ = -2 \ln |u|^{\sin^2 x} + \ln x + C_1 \]
\[ = -2 \ln |\sin x^2| + \ln x + C_1 \]
\[ = \ln(\sin x^2)^{-2} + \ln x + C_1 \]

Exponentiate both sides.

\[
c'(x) = e^{\ln(\sin x^2)^{-2} + \ln x + C_1}
\]
\[ = e^{\ln(\sin x^2)^{-2}} e^{\ln x + C_1} \]
\[ = (\sin x^2)^{-2} x e^{C_1} \]
\[ = \frac{x}{\sin^2 x^2} e^{C_1} \]

Integrate both sides with respect to \( x \) once more.

\[
c(x) = \int^x \frac{s}{\sin^2 s^2} e^{C_1} ds + C_2 \]

Make the substitution,
\[ v = s^2 \]
\[ dv = 2s ds \quad \rightarrow \quad \frac{dv}{2} = s ds. \]

As a result,

\[
c(x) = \int^x \frac{dv/2}{\sin^2 v} e^{C_1} + C_2 \]
\[ = \frac{e^{C_1}}{2} \int^x \csc^2 v dv + C_2 \]
\[ = -\frac{e^{C_1}}{2} \cot x^2 + C_2 \]
\[ = C_3 \cot x^2 + C_2, \]

where a new constant \( C_3 \) was used for \(-e^{C_1}/2\). Therefore, the general solution is

\[ y(x) = C_3 \cos x^2 + C_2 \sin x^2; \]

the second solution is \( \cos x^2 \).

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