Problem 30

In each of Problems 23 through 30, use the method of reduction of order to find a second solution of the given differential equation.

$$x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \qquad y_1(x) = x^{-1/2}\sin x$$

Solution

Because this ODE is homogeneous, any constant multiple of $y_1(x)$ is also a solution: $cy_1(x) = cx^{-1/2}\sin x$. According to the method of reduction of order, the general solution is obtained by allowing c to vary as a function of x.

$$y(x) = c(x)x^{-1/2}\sin x$$

Substitute this formula for $y(x)$ into the ODE.

$$x^2[c(x)x^{-1/2}\sin x]'' + x[c(x)x^{-1/2}\sin x]' + (x^2 - 0.25)[c(x)x^{-1/2}\sin x] = 0$$

Evaluate the derivatives using the product rule.

$$\begin{aligned} x^2 &\left[c'(x)x^{-1/2}\sin x - \frac{c(x)x^{-3/2}\sin x}{2} + c(x)x^{-1/2}\cos x\right]' \\ &+ x\left[c'(x)x^{-1/2}\sin x - \frac{c(x)x^{-3/2}\sin x}{2} + c(x)x^{-1/2}\cos x\right] + (x^2 - 0.25)c(x)x^{-1/2}\sin x = 0 \end{aligned}$$

$$\begin{aligned} x^2 &\left[c''(x)x^{-1/2}\sin x - \frac{c'(x)x^{-3/2}\sin x}{2} + c'(x)x^{-1/2}\cos x \right. \\ &- \frac{c'(x)x^{-3/2}\sin x}{2} + \frac{3c(x)x^{-5/2}\sin x}{4} - \frac{c(x)x^{-3/2}\cos x}{2} \\ &\left. + c'(x)x^{-1/2}\cos x - \frac{c(x)x^{-3/2}\cos x}{2} - c(x)x^{-1/2}\sin x\right] \\ &+ x\left[c'(x)x^{-1/2}\sin x - \frac{c(x)x^{-3/2}\sin x}{2} + c(x)x^{-1/2}\cos x\right] \\ &+ (x^2 - 0.25)c(x)x^{-1/2}\sin x = 0 \end{aligned}$$

$$\begin{aligned} c''(x)&x^{3/2}\sin x - \cancel{\frac{c'(x)x^{1/2}\sin x}{2}} + c'(x)x^{3/2}\cos x \\ &- \cancel{\frac{c'(x)x^{1/2}\sin x}{2}} + \cancel{\frac{3c(x)x^{-1/2}\sin x}{4}} - \frac{c(x)x^{1/2}\cos x}{2} \\ &+ c'(x)x^{3/2}\cos x - \frac{c(x)x^{1/2}\cos x}{2} - \cancel{c(x)x^{3/2}\sin x} \\ &+ \cancel{c'(x)x^{1/2}}\sin x - \cancel{\frac{c(x)x^{-1/2}\sin x}{2}} + c(x)x^{1/2}\cos x \\ &+ \cancel{c(x)x^{3/2}\sin x} - \cancel{0.25c(x)x^{-1/2}\sin x} = 0 \end{aligned}$$

Simplifying, we obtain the following ODE for $c(x)$.

$$c''(x)x^{3/2}\sin x + 2c'(x)x^{3/2}\cos x - \frac{c(x)x^{1/2}\cos x}{2} - \frac{c(x)x^{1/2}\cos x}{2} + c(x)x^{1/2}\cos x = 0$$

$$c''(x)x^{3/2}\sin x + 2c'(x)x^{3/2}\cos x = 0$$

Divide both sides by $x^{3/2}\sin x$.

$$c''(x) + 2\frac{\cos x}{\sin x}c'(x) = 0$$

Solve for $c''(x)/c'(x)$.

$$\frac{c''(x)}{c'(x)} = -2\frac{\cos x}{\sin x}$$

The left side can be written as $d/dx[\ln c'(x)]$ by the chain rule.

$$\frac{d}{dx}[\ln c'(x)] = -2\cot x$$

Integrate both sides with respect to x.

$$\begin{aligned}
\ln c'(x) &= \int^x -2\cot r\, dr + C_1 \\
&= -2\int^x \cot r\, dr + C_1 \\
&= -2\ln|\sin x| + C_1 \\
&= \ln(\sin x)^{-2} + C_1
\end{aligned}$$

Exponentiate both sides.

$$\begin{aligned}
c'(x) &= e^{\ln(\sin x)^{-2} + C_1} \\
&= e^{\ln(\sin x)^{-2}}e^{C_1} \\
&= (\sin x)^{-2}e^{C_1} \\
&= e^{C_1}\csc^2 x
\end{aligned}$$

Integrate both sides with respect to x once more.

$$\begin{aligned}
c(x) &= \int^x e^{C_1}\csc^2 r\, dr + C_2 \\
&= -e^{C_1}\cot x + C_2
\end{aligned}$$

Use a new constant C_3 for $-e^{C_1}$.

$$c(x) = C_3\cot x + C_2$$

The general solution is then

$$\begin{aligned}
y(x) &= c(x)x^{-1/2}\sin x \\
&= C_3x^{-1/2}\cos x + C_2x^{-1/2}\sin x,
\end{aligned}$$

which means the second solution is $y_2(x) = x^{-1/2}\cos x$.