Problem 31

The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution, and then find the general solution in the form of an integral.

Solution

Check that $y_1(x) = \exp(-\delta x^2/2)$ is a solution to the ODE.

$$y_1'' + \delta(xy_1' + y_1) \stackrel{?}{=} 0$$

$$[\exp(-\delta x^2/2)]'' + \delta\{x[\exp(-\delta x^2/2)]' + \exp(-\delta x^2/2)\} \stackrel{?}{=} 0$$

$$[-\delta x \exp(-\delta x^2/2)]' + \delta\{x[-\delta x \exp(-\delta x^2/2)] + \exp(-\delta x^2/2)\} \stackrel{?}{=} 0$$

$$[-\delta \exp(-\delta x^2/2) + \delta^2 x^2 \exp(-\delta x^2/2)] + \delta\{x[-\delta x \exp(-\delta x^2/2)] + \exp(-\delta x^2/2)\} \stackrel{?}{=} 0$$

$$-\cancel{\delta \exp(-\delta x^2/2)} + \cancel{\delta^2 x^2 \exp(-\delta x^2/2)} - \cancel{\delta^2 x^2 \exp(-\delta x^2/2)} + \cancel{\delta \exp(-\delta x^2/2)} \stackrel{?}{=} 0$$

$$0 = 0$$

Since $y_1(x)$ is a solution, the general solution is of the form $c(x)y_1(x)$ by the method of reduction of order. Substitute it into the ODE and solve for $c(x)$.

$$[c(x)\exp(-\delta x^2/2)]'' + \delta\{x[c(x)\exp(-\delta x^2/2)]' + c(x)\exp(-\delta x^2/2)\} = 0$$

Evaluate the derivatives using the product rule.

$$\begin{aligned}&[c'(x)\exp(-\delta x^2/2) - c(x)\delta x \exp(-\delta x^2/2)]' \\ &\qquad + \delta\{x[c'(x)\exp(-\delta x^2/2) - c(x)\delta x\exp(-\delta x^2/2)] + c(x)\exp(-\delta x^2/2)\} = 0\end{aligned}$$

$$\begin{aligned}&[c''(x)\exp(-\delta x^2/2) - c'(x)\delta x\exp(-\delta x^2/2) - c'(x)\delta x \exp(-\delta x^2/2) \\ &\quad - c'(x)\delta\exp(-\delta x^2/2) + c(x)\delta^2 x^2\exp(-\delta x^2/2)] \\ &\qquad + \delta\{x[c'(x)\exp(-\delta x^2/2) - c(x)\delta x\exp(-\delta x^2/2)] + c(x)\exp(-\delta x^2/2)\} = 0\end{aligned}$$

$$\begin{aligned}&c''(x)\exp(-\delta x^2/2) - c'(x)\delta x\exp(-\delta x^2/2) - \cancel{c'(x)\delta x\exp(-\delta x^2/2)} \\ &\quad - \cancel{c'(x)\delta\exp(-\delta x^2/2)} + \cancel{c(x)\delta^2x^2\exp(-\delta x^2/2)} \\ &\qquad + \cancel{c'(x)\delta x\exp(-\delta x^2/2)} - \cancel{c(x)\delta^2 x^2\exp(-\delta x^2/2)} + \cancel{c(x)\delta\exp(-\delta x^2/2)} = 0\end{aligned}$$

$$c''(x)\exp(-\delta x^2/2) - c'(x)\delta x\exp(-\delta x^2/2) = 0$$

Multiply both sides by $\exp(\delta x^2/2)$.

$$c''(x) - \delta x c'(x) = 0$$

This is a linear first-order ODE for $c'(x)$, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left[\int^x (-\delta r)\,dr\right] = e^{-\delta x^2/2}$$

Proceed with the multiplication.

$$e^{-\delta x^2/2}c''(x) - \delta x e^{-\delta x^2/2}c'(x) = 0$$

The left side can be written as $d/dx[Ic'(x)]$ by the product rule.

$$\frac{d}{dx}[e^{-\delta x^2/2}c'(x)] = 0$$

Integrate both sides with respect to x.

$$e^{-\delta x^2/2}c'(x) = C_1$$

Multiply both sides by $e^{\delta x^2/2}$.

$$c'(x) = C_1 e^{\delta x^2/2}$$

Integrate both sides with respect to x once more.

$$c(x) = \int^x C_1 e^{\delta r^2/2}\,dr + C_2$$

Therefore, the general solution is

$$\begin{aligned}
y(x) &= c(x)\exp(-\delta x^2/2) \\
&= \exp(-\delta x^2/2)\int^x C_1 e^{\delta r^2/2}\,dr + C_2\exp(-\delta x^2/2) \\
&= C_1\int^x e^{\delta r^2/2 - \delta x^2/2}\,dr + C_2\exp(-\delta x^2/2) \\
&= C_1\int^x e^{\delta(r^2 - x^2)/2}\,dr + C_2 e^{-\delta x^2/2}.
\end{aligned}$$

Note that the lower limit of integration is arbitrary.