Problem 32

The method of Problem 20 can be extended to second order equations with variable coefficients. If \( y_1 \) is a known nonvanishing solution of \( y'' + p(t)y' + q(t)y = 0 \), show that a second solution \( y_2 \) satisfies \( (y_2/y_1)' = W(y_1, y_2)/y_1^2 \), where \( W(y_1, y_2) \) is the Wronskian of \( y_1 \) and \( y_2 \). Then use Abel’s formula [Eq. (23) of Section 3.2] to determine \( y_2 \).

Solution

The Wronskian of \( y_1 \) and \( y_2 \) is defined as

\[
W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \quad \Rightarrow \quad W'(y_1, y_2) = y_1y_2'' + y_1'y_2' - y_1'y_2 - y_1''y_2 = y_1y_2'' - y_1'y_2.
\]

Because \( y_1 \) and \( y_2 \) are both solutions to the ODE, they satisfy

\[
y_1'' + p(t)y_1' + q(t)y_1 = 0 \\
y_2'' + p(t)y_2' + q(t)y_2 = 0.
\]

Multiply both sides of the first equation by \(-y_2\) and both sides of the second equation by \(y_1\).

\[
-y_1''y_2 - p(t)y_1'y_2 - q(t)y_1y_2 = 0 \\
y_1y_2'' + p(t)y_1'y_2 + q(t)y_1y_2 = 0
\]

Add the respective sides of each equation.

\[
y_1y_2'' - y_1''y_2 + p(t)y_1y_2' - p(t)y_1'y_2 = 0
\]

Factor \( p(t) \).

\[
y_1y_2'' - y_1''y_2 + p(t)(y_1y_2' - y_1'y_2) = 0
\]

This equation can be written in terms of the Wronskian as

\[
W' + p(t)W = 0.
\]

Solve it now.

\[
W' = -p(t)W \\
\frac{d}{dt} \ln W = -p(t) \\
\ln W = - \int^t p(s) \, ds \\
W(t) = \exp \left[ - \int^t p(s) \, ds \right]
\]

Now that the Wronskian is known, use its definition to obtain an ODE for \( y_2 \).

\[
y_1y_2' - y_1'y_2 = \exp \left[ - \int^t p(s) \, ds \right]
\]

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Divide both sides by $y_1$.

$$y'_{2} - \frac{y'_1}{y_1} y_2 = \frac{1}{y_1} \exp \left[ - \int_{t}^{t} p(s) \, ds \right]$$

Write the coefficient of $y_2$ as the derivative of a logarithm; this is possible by the chain rule.

$$y'_{2} - \frac{d}{dt} (\ln y_1) y_2 = \frac{1}{y_1} \exp \left[ - \int_{t}^{t} p(s) \, ds \right]$$

This is a first-order linear inhomogeneous ODE for $y_2(t)$, so it can be solved by multiplying both sides by an integrating factor $I$.

$$I = \exp \left( \int_{t}^{t} - \frac{d}{dr} [\ln y_1(r)] \, dr \right) = e^{-\ln y_1(t)} = e^{\ln y_1(t)^{-1}} = \frac{1}{y_1(t)}$$

Proceed with the multiplication.

$$\frac{1}{y_1(t)} y'_{2} - \frac{1}{y_1(t)} \frac{d}{dt} (\ln y_1) y_2 = \frac{1}{y_1(t)^2} \exp \left[ - \int_{t}^{t} p(s) \, ds \right]$$

The left side can be written as $d/dt (Iy_2)$ by the product rule.

$$\frac{d}{dt} \left[ \frac{1}{y_1(t)} y_2 \right] = \frac{1}{y_1(t)^2} \exp \left[ - \int_{t}^{t} p(s) \, ds \right]$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$\frac{1}{y_1(t)} y_2 = \int_{t}^{t} \frac{1}{y_1(r)^2} \exp \left[ - \int_{r}^{t} p(s) \, ds \right] \, dr$$

Therefore, multiplying both sides by $y_1(t)$,

$$y_2(t) = y_1(t) \int_{t}^{t} \frac{1}{y_1(r)^2} \exp \left[ - \int_{r}^{t} p(s) \, ds \right] \, dr.$$ 

Note that both of the lower limits of integration are arbitrary.