Problem 42

Euler Equations. In each of Problems 40 through 45, use the substitution introduced in Problem 34 in Section 3.3 to solve the given differential equation.

$$2t^2y'' - 5ty' + 5y = 0, \qquad t > 0$$

Solution

The Hard Way

Make the substitution $x = \ln t$ in the ODE. Then

$$e^x = t \quad \rightarrow \quad e^{2x} = t^2,$$

and the ODE becomes

$$2e^{2x}\frac{d^2y}{dt^2} - 5e^x\frac{dy}{dt} + 5y = 0.$$

The aim now is to find what the derivatives are in terms of this new variable by using the chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\left(\frac{1}{t}\right) = \frac{dy}{dx}\left(\frac{1}{e^x}\right) = e^{-x}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right) = \frac{1}{t}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) = \frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right)$$

Substitute these expressions into the ODE.

$$2e^{2x}\frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) - 5e^x\left(e^{-x}\frac{dy}{dx}\right) + 5y = 0$$

$$2e^x\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) - 5\frac{dy}{dx} + 5y = 0$$

$$-2\frac{dy}{dx} + 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 5y = 0$$

$$2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 5y = 0$$

As a result of making the substitution $x = \ln t$, the coefficients of the derivatives are now constant. The solution is then of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Substitute these expressions into the ODE.

$$2(r^2e^{rx}) - 7(re^{rx}) + 5(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$2r^2 - 7r + 5 = 0$$

$$(2r - 5)(r - 1) = 0$$

$$r = \left\{1, \frac{5}{2}\right\}$$

Consequently, two solutions to the ODE are $y = e^x$ and $y = e^{5x/2}$. By the principle of superposition, the general solution is a linear combination of the two.

$$y(x) = C_1 e^x + C_2 e^{5x/2}$$

Change back to the original variable with the initial substitution $x = \ln t$.

$$\begin{aligned} y(t) &= C_1 e^{\ln t} + C_2 e^{5(\ln t)/2} \\ &= C_1 t + C_2 e^{\ln t^{5/2}} \\ &= C_1 t + C_2 t^{5/2} \end{aligned}$$

The Easy Way

$$2t^2 y'' - 5ty' + 5y = 0, \qquad t > 0$$

Since this is an Euler (or equidimensional) equation, the solution is of the form $y = t^r$.

$$y = t^r \quad \rightarrow \quad y' = rt^{r-1} \quad \rightarrow \quad y'' = r(r-1)t^{r-2}$$

Substitute these expressions into the ODE.

$$2t^2[r(r-1)t^{r-2}] - 5t[rt^{r-1}] + 5t^r = 0$$

$$2r(r-1)t^r - 5rt^r + 5t^r = 0$$

Divide both sides by t^r.

$$2r(r-1) - 5r + 5 = 0$$

$$2r^2 - 7r + 5 = 0$$

$$(2r-5)(r-1) = 0$$

$$r = \left\{1, \frac{5}{2}\right\}$$

Consequently, two solutions to the ODE are $y = t^1$ and $y = t^{5/2}$. By the principle of superposition, the general solution is a linear combination of the two.

$$y(t) = C_3 t + C_4 t^{5/2}$$