Problem 43

Euler Equations. In each of Problems 40 through 45, use the substitution introduced in Problem 34 in Section 3.3 to solve the given differential equation.

$$t^2y'' + 3ty' + y = 0, \qquad t > 0$$

Solution

The Hard Way

Make the substitution $x = \ln t$ in the ODE. Then

$$e^x = t \quad \rightarrow \quad e^{2x} = t^2,$$

and the ODE becomes

$$e^{2x}\frac{d^2y}{dt^2} + 3e^x\frac{dy}{dt} + y = 0.$$

The aim now is to find what the derivatives are in terms of this new variable by using the chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\left(\frac{1}{t}\right) = \frac{dy}{dx}\left(\frac{1}{e^x}\right) = e^{-x}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right) = \frac{1}{t}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) = \frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right)$$

Substitute these expressions into the ODE.

$$e^{2x}\frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + 3e^x\left(e^{-x}\frac{dy}{dx}\right) + y = 0$$

$$e^x\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + 3\frac{dy}{dx} + y = 0$$

$$-\frac{dy}{dx} + \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \tag{1}$$

As a result of making the substitution $x = \ln t$, the coefficients of the derivatives are now constant. The solution is then of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Substitute these expressions into the ODE.

$$r^2e^{rx} + 2(re^{rx}) + e^{rx} = 0$$

Divide both sides by e^{rx}.

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$$r = \{-1\}$$

Consequently, one solution to the ODE is $y = e^{-x}$. Use the method of reduction of order here to find the general solution: Plug $y(x) = c(x)e^{-x}$ into equation (1).

$$\frac{d^2}{dx^2}[c(x)e^{-x}] + 2\frac{d}{dx}[c(x)e^{-x}] + [c(x)e^{-x}] = 0$$

Evaluate the derivatives using the product rule.

$$\frac{d}{dx}[c'(x)e^{-x} - c(x)e^{-x}] + 2[c'(x)e^{-x} - c(x)e^{-x}] + [c(x)e^{-x}] = 0$$

$$[c''(x)e^{-x} - c'(x)e^{-x} - c'(x)e^{-x} + c(x)e^{-x}] + 2[c'(x)e^{-x} - c(x)e^{-x}] + [c(x)e^{-x}] = 0$$

$$c''(x)e^{-x} - \cancel{c'(x)e^{-x}} - \cancel{c'(x)e^{-x}} + \cancel{c(x)e^{-x}} + \cancel{2c'(x)e^{-x}} - \cancel{2c(x)e^{-x}} + \cancel{c(x)e^{-x}} = 0$$

$$c''(x)e^{-x} = 0$$

Multiply both sides by e^x.

$$c''(x) = 0$$

Integrate both sides with respect to x.

$$c'(x) = C_1$$

Integrate both sides with respect to x once more.

$$c(x) = C_1x + C_2$$

Since $y(x) = c(x)e^{-x}$, the general solution is

$$y(x) = C_1xe^{-x} + C_2e^{-x}.$$

Finally, change back to the original variable with the initial substitution $x = \ln t$.

$$\begin{aligned} y(t) &= C_1(\ln t)e^{-\ln t} + C_2e^{-\ln t} \\ &= C_1(\ln t)e^{\ln t^{-1}} + C_2e^{\ln t^{-1}} \\ &= C_1t^{-1}\ln t + C_2t^{-1} \end{aligned}$$

The Easy Way

$$t^2y'' + 3ty' + y = 0, \qquad t > 0$$

Since this is an Euler (or equidimensional) equation, the solution is of the form $y = t^r$.

$$y = t^r \quad \rightarrow \quad y' = rt^{r-1} \quad \rightarrow \quad y'' = r(r-1)t^{r-2}$$

Substitute these expressions into the ODE.

$$t^2[r(r-1)t^{r-2}] + 3t[rt^{r-1}] + t^r = 0$$

$$r(r-1)t^r + 3rt^r + t^r = 0$$

Divide both sides by t^r.

$$r(r-1) + 3r + 1 = 0$$

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$$r = \{-1\}$$

One solution to the ODE is then t^{-1}. The ODE is homogeneous, so any constant multiple of this, $y = ct^{-1}$, is also a solution. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t: $y(t) = c(t)t^{-1}$. Substitute this into the original ODE to find what $c(t)$ is.

$$t^2[c(t)t^{-1}]'' + 3t[c(t)t^{-1}]' + [c(t)t^{-1}] = 0$$

Evaluate the derivatives by using the product rule.

$$t^2[c'(t)t^{-1} - c(t)t^{-2}]' + 3t[c'(t)t^{-1} - c(t)t^{-2}] + [c(t)t^{-1}] = 0$$

$$t^2[c''(t)t^{-1} - c'(t)t^{-2} - c'(t)t^{-2} + 2c(t)t^{-3}] + 3t[c'(t)t^{-1} - c(t)t^{-2}] + [c(t)t^{-1}] = 0$$

$$c''(t)t - c'(t) - c'(t) + \cancel{2c(t)t^{-1}} + 3c'(t) - \cancel{3c(t)t^{-1}} + \cancel{c(t)t^{-1}} = 0$$

$$c''(t)t + c'(t) = 0$$

Solve for $c''(t)/c'(t)$.

$$\frac{c''(t)}{c'(t)} = -\frac{1}{t}$$

The left side can be written as $d/dt[\ln c'(t)]$ by the chain rule.

$$\frac{d}{dt}[\ln c'(t)] = -\frac{1}{t}$$

Integrate both sides with respect to t.

$$\ln c'(t) = -\ln t + C_3$$

Exponentiate both sides.

$$\begin{aligned} c'(t) &= e^{-\ln t + C_3} \\ &= e^{\ln t^{-1} + C_3} \\ &= e^{\ln t^{-1}}e^{C_3} \\ &= t^{-1}e^{C_3} \end{aligned}$$

Integrate both sides with respect to t once more.

$$c(t) = e^{C_3} \ln t + C_4$$

Since the general solution is $y(t) = c(t)t^{-1}$, we have

$$y(t) = e^{C_3} t^{-1} \ln t + C_4 t^{-1}.$$

Therefore, using a new constant C_5 for e^{C_3},

$$y(t) = C_5 t^{-1} \ln t + C_4 t^{-1}.$$