Problem 43

Euler Equations. In each of Problems 40 through 45, use the substitution introduced in Problem 34 in Section 3.3 to solve the given differential equation.

\[ t^2 y'' + 3ty' + y = 0, \quad t > 0 \]

Solution

The Hard Way

Make the substitution \( x = \ln t \) in the ODE. Then

\[ e^x = t \quad \Rightarrow \quad e^{2x} = t^2, \]

and the ODE becomes

\[ e^{2x} \frac{d^2 y}{dt^2} + 3e^x \frac{dy}{dt} + y = 0. \]

The aim now is to find what the derivatives are in terms of this new variable by using the chain rule.

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{dy}{dx} \left( \frac{1}{t} \right) = \frac{dy}{dx} \left( \frac{1}{e^x} \right) = e^{-x} \frac{dy}{dx}
\]

\[
\frac{d^2 y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = \frac{dx}{dt} \frac{d}{dx} \left( e^{-x} \frac{dy}{dx} \right) = \frac{1}{t} \left( -e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2 y}{dx^2} \right) = \frac{1}{e^x} \left( -e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2 y}{dx^2} \right)
\]

Substitute these expressions into the ODE.

\[
e^{2x} \frac{1}{e^x} \left( -e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2 y}{dx^2} \right) + 3e^x \left( e^{-x} \frac{dy}{dx} \right) + y = 0
\]

\[
e^x \left( -e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2 y}{dx^2} \right) + 3 \frac{dy}{dx} + y = 0
\]

\[
- \frac{dy}{dx} + \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + y = 0
\]

\[
\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = 0
\]

As a result of making the substitution \( x = \ln t \), the coefficients of the derivatives are now constant. The solution is then of the form \( y = e^{rx} \).

\[ y = e^{rx} \quad \Rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \Rightarrow \quad \frac{d^2 y}{dx^2} = r^2 e^{rx} \]

Substitute these expressions into the ODE.

\[ r^2 e^{rx} + 2(re^{rx}) + e^{rx} = 0 \]

Divide both sides by \( e^{rx} \).

\[ r^2 + 2r + 1 = 0 \]

\[ (r + 1)^2 = 0 \]

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Consequently, one solution to the ODE is $y = e^{-x}$. Use the method of reduction of order here to find the general solution: Plug $y(x) = c(x)e^{-x}$ into equation (1).

$$\frac{d^2}{dx^2}[c(x)e^{-x}] + 2\frac{d}{dx}[c(x)e^{-x}] + [c(x)e^{-x}] = 0$$

Evaluate the derivatives using the product rule.

$$\frac{d}{dx}[c'(x)e^{-x} - c(x)e^{-x}] + 2[c'(x)e^{-x} - c(x)e^{-x}] + [c(x)e^{-x}] = 0$$

$$c''(x)e^{-x} - c'(x)e^{-x} - c'(x)e^{-x} + c(x)e^{-x} + 2[c'(x)e^{-x} - c(x)e^{-x}] + [c(x)e^{-x}] = 0$$

$$c''(x)e^{-x} = 0$$

Multiply both sides by $e^x$.

$$c''(x) = 0$$

Integrate both sides with respect to $x$.

$$c'(x) = C_1$$

Integrate both sides with respect to $x$ once more.

$$c(x) = C_1x + C_2$$

Since $y(x) = c(x)e^{-x}$, the general solution is

$$y(x) = C_1xe^{-x} + C_2e^{-x}.$$  

Finally, change back to the original variable with the initial substitution $x = \ln t$.

$$y(t) = C_1(\ln t)e^{-\ln t} + C_2e^{-\ln t}$$

$$= C_1(\ln t)t^{1-1} + C_2t^{1-1}$$

$$= C_1t^{-1} \ln t + C_2t^{-1}$$
The Easy Way

\[ t^2 y'' + 3ty' + y = 0, \quad t > 0 \]

Since this is an Euler (or equidimensional) equation, the solution is of the form \( y = t^r \).

\[ y = t^r \rightarrow y' = rt^{r-1} \rightarrow y'' = r(r-1)t^{r-2} \]

Substitute these expressions into the ODE.

\[ t^2[r(r-1)t^{r-2}] + 3t[rt^{r-1}] + t^r = 0 \]
\[ r(r-1)t^r + 3rt^r + t^r = 0 \]

Divide both sides by \( t^r \).

\[ r(r-1) + 3r + 1 = 0 \]
\[ r^2 + 2r + 1 = 0 \]
\[ (r+1)^2 = 0 \]
\[ r = \{ -1 \} \]

One solution to the ODE is then \( t^{-1} \). The ODE is homogeneous, so any constant multiple of this, \( y = ct^{-1} \), is also a solution. According to the method of reduction of order, the general solution is found by allowing \( c \) to vary as a function of \( t \): \( y(t) = c(t)t^{-1} \). Substitute this into the original ODE to find what \( c(t) \) is.

\[ t^2[c(t)t^{-1}]'' + 3t[c(t)t^{-1}]' + [c(t)t^{-1}] = 0 \]

Evaluate the derivatives by using the product rule.

\[ t^2[c'(t)t^{-1} - c(t)t^{-2}]' + 3t[c'(t)t^{-1} - c(t)t^{-2}] + [c(t)t^{-1}] = 0 \]
\[ t^2[c''(t)t^{-1} - c'(t)t^{-2} - c(t)t^{-3} + 2c(t)t^{-3}] + 3t[c'(t)t^{-1} - c(t)t^{-2}] + [c(t)t^{-1}] = 0 \]
\[ c''(t)t - c'(t) - c'(t) + 2c(t)t^{-1} + 3c'(t) - 3c(t)t^{-1} + [c(t)t^{-1}] = 0 \]
\[ c''(t)t + c'(t) = 0 \]

Solve for \( c''(t)/c'(t) \).

\[ \frac{c''(t)}{c'(t)} = -\frac{1}{t} \]

The left side can be written as \( \frac{d}{dt}[\ln c'(t)] \) by the chain rule.

\[ \frac{d}{dt}[\ln c'(t)] = -\frac{1}{t} \]

Integrate both sides with respect to \( t \).

\[ \ln c'(t) = -\ln t + C_3 \]

Exponentiate both sides.

\[ c'(t) = e^{-\ln t + C_3} \]
\[ = e^{\ln t^{-1} + C_3} \]
\[ = e^{\ln t^{-1}}e^{C_3} \]
\[ = t^{-1}e^{C_3} \]
Integrate both sides with respect to $t$ once more.

$$c(t) = e^{C_3} \ln t + C_4$$

Since the general solution is $y(t) = c(t)t^{-1}$, we have

$$y(t) = e^{C_3}t^{-1} \ln t + C_4t^{-1}.$$ 

Therefore, using a new constant $C_5$ for $e^{C_3}$,

$$y(t) = C_5t^{-1} \ln t + C_4t^{-1}.$$