Problem 45

Euler Equations. In each of Problems 40 through 45, use the substitution introduced in Problem 34 in Section 3.3 to solve the given differential equation.

$$t^2y'' + 5ty' + 13y = 0, \qquad t > 0$$

Solution

The Hard Way

Make the substitution $x = \ln t$ in the ODE. Then

$$e^x = t \quad \to \quad e^{2x} = t^2,$$

and the ODE becomes

$$e^{2x}\frac{d^2y}{dt^2} + 5e^x\frac{dy}{dt} + 13y = 0.$$

The aim now is to find what the derivatives are in terms of this new variable by using the chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\left(\frac{1}{t}\right) = \frac{dy}{dx}\left(\frac{1}{e^x}\right) = e^{-x}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right) = \frac{1}{t}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) = \frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right)$$

Substitute these expressions into the ODE.

$$e^{2x}\frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + 5e^x\left(e^{-x}\frac{dy}{dx}\right) + 13y = 0$$

$$e^x\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + 5\frac{dy}{dx} + 13y = 0$$

$$-\frac{dy}{dx} + \frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 13y = 0$$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0 \tag{1}$$

As a result of making the substitution $x = \ln t$, the coefficients of the derivatives are now constant. The solution is then of the form $y = e^{rx}$.

$$y = e^{rx} \quad \to \quad \frac{dy}{dx} = re^{rx} \quad \to \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Substitute these expressions into the ODE.

$$r^2e^{rx} + 4(re^{rx}) + 13(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 4r + 13 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4(1)(13)}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

$$r = \{-2 - 3i, -2 + 3i\}$$

Consequently, two solutions to the ODE are $y = e^{(-2-3i)x}$ and $y = e^{(-2+3i)x}$. By the principle of superposition, the general solution is a linear combination of the two.

$$\begin{aligned}
y(x) &= C_1 e^{(-2-3i)x} + C_2 e^{(-2+3i)x} \\
&= C_1 e^{-2x-3ix} + C_2 e^{-2x+3ix} \\
&= C_1 e^{-2x} e^{-3ix} + C_2 e^{-2x} e^{3ix} \\
&= C_1 e^{-2x}[\cos(-3x) + i\sin(-3x)] + C_2 e^{-2x}[\cos(3x) + i\sin(3x)] \\
&= C_1 e^{-2x}[\cos(3x) - i\sin(3x)] + C_2 e^{-2x}[\cos(3x) + i\sin(3x)] \\
&= C_1 e^{-2x}\cos 3x - iC_1 e^{-2x}\sin 3x + C_2 e^{-2x}\cos 3x + iC_2 e^{-2x}\sin 3x \\
&= (C_1 + C_2)e^{-2x}\cos 3x + (-iC_1 + iC_2)e^{-2x}\sin 3x
\end{aligned}$$

Use the new variables, C_3 and C_4, for $C_1 + C_2$ and $-iC_1 + iC_2$, respectively.

$$y(x) = C_3 e^{-2x}\cos 3x + C_4 e^{-2x}\sin 3x$$

Finally, change back to the original variable with the initial substitution $x = \ln t$.

$$\begin{aligned}
y(t) &= C_3 e^{-2\ln t}\cos(3\ln t) + C_4 e^{-2\ln t}\sin(3\ln t) \\
&= C_3 e^{\ln t^{-2}}\cos(3\ln t) + C_4 e^{\ln t^{-2}}\sin(3\ln t) \\
&= C_3 t^{-2}\cos(3\ln t) + C_4 t^{-2}\sin(3\ln t)
\end{aligned}$$

The Easy Way

$$t^2y'' + 5ty' + 13y = 0, \qquad t > 0$$

Since this is an Euler (or equidimensional) equation, the solution is of the form $y = t^r$.

$$y = t^r \quad \to \quad y' = rt^{r-1} \quad \to \quad y'' = r(r-1)t^{r-2}$$

Substitute these expressions into the ODE.

$$t^2[r(r-1)t^{r-2}] + 5t[rt^{r-1}] + 13(t^r) = 0$$

$$r(r-1)t^r + 5rt^r + 13(t^r) = 0$$

Divide both sides by t^r.

$$r(r-1) + 5r + 13 = 0$$

$$r^2 + 4r + 13 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4(1)(13)}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

$$r = \{-2 - 3i, -2 + 3i\}$$

Consequently, two solutions to the ODE are $y = t^{-2-3i}$ and $y = t^{-2+3i}$. By the principle of superposition, the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_5 t^{-2-3i} + C_6 t^{-2+3i} \\
&= C_5 t^{-2} t^{-3i} + C_6 t^{-2} t^{3i} \\
&= C_5 t^{-2} e^{\ln t^{-3i}} + C_6 t^{-2} e^{\ln t^{3i}} \\
&= C_5 t^{-2} e^{(-3i)\ln t} + C_6 t^{-2} e^{(3i)\ln t} \\
&= C_5 t^{-2}[\cos(-3\ln t) + i\sin(-3\ln t)] + C_6 t^{-2}[\cos(3\ln t) + i\sin(3\ln t)] \\
&= C_5 t^{-2}[\cos(3\ln t) - i\sin(3\ln t)] + C_6 t^{-2}[\cos(3\ln t) + i\sin(3\ln t)] \\
&= C_5 t^{-2}\cos(3\ln t) - iC_5 t^{-2}\sin(3\ln t) + C_6 t^{-2}\cos(3\ln t) + iC_6 t^{-2}\sin(3\ln t) \\
&= (C_5 + C_6) t^{-2}\cos(3\ln t) + (-iC_5 + iC_6) t^{-2}\sin(3\ln t)
\end{aligned}$$

Use the new variables, C_7 and C_8, for $C_5 + C_6$ and $-iC_5 + iC_6$, respectively.

$$y(t) = C_7 t^{-2}\cos(3\ln t) + C_8 t^{-2}\sin(3\ln t)$$