Problem 3

In each of Problems 1 through 14, find the general solution of the given differential equation.

\[ y'' - y' - 2y = -2t + 4t^2 \]

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y'' - y' - 2y = 0 \tag{1} \]

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \rightarrow y'_c = re^{rt} \rightarrow y''_c = r^2e^{rt} \]

Substitute these expressions into the ODE.

\[ r^2e^{rt} - re^{rt} - 2(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 - r - 2 = 0 \]

\[ (r - 2)(r + 1) = 0 \]

\[ r = \{ -1, 2 \} \]

Two solutions to equation (1) are then \( y_c = e^{-t} \) and \( y_c = e^{2t} \). By the principle of superposition, the general solution is a linear combination of these two.

\[ y_c(t) = C_1e^{-t} + C_2e^{2t} \]

The particular solution satisfies

\[ y''_p - y'_p - 2y_p = -2t + 4t^2. \]

Since the inhomogeneous term is a polynomial, we assume the solution is of the form \( y_p(t) = A + Bt + Ct^2 \). Substitute this into the ODE to determine \( A \).

\[
\begin{align*}
(A + Bt + Ct^2)'' - (A + Bt + Ct^2)' - 2(A + Bt + Ct^2) &= -2t + 4t^2 \\
(B + 2Ct)' - (B + 2Ct) - 2(A + Bt + Ct^2) &= -2t + 4t^2 \\
(2C) - (B + 2Ct) - 2(A + Bt + Ct^2) &= -2t + 4t^2 \\
2C - B - 2Ct - 2A - 2Bt - 2Ct^2 &= -2t + 4t^2 \\
(-2A - B + 2C) + (-2B - 2C)t + (-2C)t^2 &= -2t + 4t^2
\end{align*}
\]
For this equation to be true, $A$ and $B$ and $C$ must satisfy the following system of equations.

$$-2A - B + 2C = 0$$
$$-2B - 2C = -2$$
$$-2C = 4$$

Solving it yields $A = -7/2$ and $B = 3$ and $C = -2$, which means

$$y_p(t) = -\frac{7}{2} + 3t - 2t^2.$$ 

Therefore,

$$y(t) = C_1e^{-t} + C_2e^{2t} - \frac{7}{2} + 3t - 2t^2.$$