Problem 6

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$y'' + 2y' = 3 + 4\sin 2t$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 2y_c' = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 2(re^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2r = 0$$

$$r(r+2) = 0$$

$$r = \{-2, 0\}$$

Two solutions to equation (1) are then $y_c = e^{-2t}$ and $y_c = e^0 = 1$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1 e^{-2t} + C_2$$

The particular solution satisfies

$$y_p'' + 2y_p' = 3 + 4\sin 2t.$$

The inhomogeneous term has two components, a constant and a sine function. Since $y_c(t)$ has the arbitrary constant C_2, an extra factor of t needs to multiply the constant in the trial solution. Also, because both odd and even derivatives are present, both sine and cosine need to be included. The trial solution is thus $y_p(t) = At + B\cos 2t + C\sin 2t$. Substitute this into the ODE to determine A and B and C.

$$(At + B\cos 2t + C\sin 2t)'' + 2(At + B\cos 2t + C\sin 2t)' = 3 + 4\sin 2t$$

$$(A - 2B\sin 2t + 2C\cos 2t)' + 2(A - 2B\sin 2t + 2C\cos 2t) = 3 + 4\sin 2t$$

$$(-4B\cos 2t - 4C\sin 2t) + 2(A - 2B\sin 2t + 2C\cos 2t) = 3 + 4\sin 2t$$

$$-4B\cos 2t - 4C\sin 2t + 2A - 4B\sin 2t + 4C\cos 2t = 3 + 4\sin 2t$$

$$2A + (-4B + 4C)\cos 2t + (-4B - 4C)\sin 2t = 3 + 4\sin 2t$$

For this equation to be true, A and B and C must satisfy the following system of equations.

$$\begin{aligned} 2A &= 3 \\ -4B + 4C &= 0 \\ -4B - 4C &= 4 \end{aligned}$$

Solving it yields $A = 3/2$ and $B = -1/2$ and $C = -1/2$, which means

$$y_p(t) = \frac{3}{2}t - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t.$$

Therefore,

$$y(t) = C_1 e^{-2t} + C_2 + \frac{3}{2}t - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t.$$