Problem 8

In each of Problems 1 through 14, find the general solution of the given differential equation.

\[ y'' + 2y' + y = 2e^{-t} \]

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y'' + 2y' + y = 0 \quad (1) \]

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \quad \rightarrow \quad y'_c = re^{rt} \quad \rightarrow \quad y''_c = r^2 e^{rt} \]

Substitute these expressions into the ODE.

\[ r^2 e^{rt} + 2(re^{rt}) + e^{rt} = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 + 2r + 1 = 0 \]

\[ (r + 1)^2 = 0 \]

\[ r = \{-1\} \]

One solution to equation (1) is then \( y_c = e^{-t} \). Use the method of reduction of order to find the general solution: Plug in \( c(t)e^{-t} \) into equation (1) to obtain an ODE for \( c(t) \).

\[

c''(t)e^{-t} - c'(t)e^{-t} - c'(t)e^{-t} + c(t)e^{-t} + 2c'(t)e^{-t} - 2c(t)e^{-t} + c(t)e^{-t} + c(t)e^{-t} = 0
\]

\[ c''(t)e^{-t} = 0 \]

Multiply both sides by \( e^t \).

\[ c''(t) = 0 \]

Integrate both sides with respect to \( t \).

\[ c'(t) = C_1 \]

Integrate both sides with respect to \( t \) once more.

\[ c(t) = C_1 t + C_2 \]

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The complementary solution is then

\[ y_c(t) = C_1 te^{-t} + C_2 e^{-t}. \]

On the other hand, the particular solution satisfies

\[ y_p'' + 2y_p' + y_p = 2e^{-t}. \]

We would use the trial solution, \( Ae^{-t} \), but because \( e^{-t} \) is already a solution of equation (1), we will use \( Ae^{-t} \). Actually, since \( te^{-t} \) also satisfies equation (1), we will use \( y_p(t) = At^2 e^{-t} \).

Substitute this into the ODE to determine \( A \) and \( B \).

\[
(At^2 e^{-t})'' + 2(At^2 e^{-t})' + At^2 e^{-t} = 2e^{-t} \\
(2Ate^{-t} - At^2 e^{-t})' + 2(2Ate^{-t} - At^2 e^{-t}) + At^2 e^{-t} = 2e^{-t} \\
(2Ae^{-t} - 2Ate^{-t} + At^2 e^{-t}) + 2(2Ate^{-t} - At^2 e^{-t}) + At^2 e^{-t} = 2e^{-t} \\
2Ae^{-t} - 4Ate^{-t} + At^2 e^{-t} + 4Ate^{-t} - 2At^2 e^{-t} + At^2 e^{-t} = 2e^{-t} \\
2Ae^{-t} = 2e^{-t}
\]

For this equation to be true, \( A \) must satisfy

\[ 2A = 2, \]

or \( A = 1. \)

\[ y_p(t) = t^2 e^{-t}. \]

Therefore,

\[ y(t) = C_1 te^{-t} + C_2 e^{-t} + t^2 e^{-t}. \]