Problem 8

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$y'' + 2y' + y = 2e^{-t}$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 2y_c' + y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \to \quad y_c' = re^{rt} \quad \to \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 2(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$$r = \{-1\}$$

One solution to equation (1) is then $y_c = e^{-t}$. Use the method of reduction of order to find the general solution: Plug in $c(t)e^{-t}$ into equation (1) to obtain an ODE for $c(t)$.

$$[c(t)e^{-t}]'' + 2[c(t)e^{-t}]' + c(t)e^{-t} = 0$$

$$[c'(t)e^{-t} - c(t)e^{-t}]' + 2[c'(t)e^{-t} - c(t)e^{-t}] + c(t)e^{-t} = 0$$

$$[c''(t)e^{-t} - c'(t)e^{-t} - c'(t)e^{-t} + c(t)e^{-t}] + 2[c'(t)e^{-t} - c(t)e^{-t}] + c(t)e^{-t} = 0$$

$$c''(t)e^{-t} - \cancel{c'(t)e^{-t}} - \cancel{c'(t)e^{-t}} + \cancel{c(t)e^{-t}} + \cancel{2c'(t)e^{-t}} - \cancel{2c(t)e^{-t}} + \cancel{c(t)e^{-t}} = 0$$

$$c''(t)e^{-t} = 0$$

Multiply both sides by e^t.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

The complementary solution is then

$$y_c(t) = C_1 te^{-t} + C_2 e^{-t}.$$

On the other hand, the particular solution satisfies

$$y_p'' + 2y_p' + y_p = 2e^{-t}.$$

We would use the trial solution, Ae^{-t}, but because e^{-t} is already a solution of equation (1), we will use Ate^{-t}. Actually, since te^{-t} also satisfies equation (1), we will use $y_p(t) = At^2e^{-t}$. Substitute this into the ODE to determine A and B.

$$(At^2e^{-t})'' + 2(At^2e^{-t})' + At^2e^{-t} = 2e^{-t}$$

$$(2Ate^{-t} - At^2e^{-t})' + 2(2Ate^{-t} - At^2e^{-t}) + At^2e^{-t} = 2e^{-t}$$

$$(2Ae^{-t} - 2Ate^{-t} - 2Ate^{-t} + At^2e^{-t}) + 2(2Ate^{-t} - At^2e^{-t}) + At^2e^{-t} = 2e^{-t}$$

$$2Ae^{-t} - \cancel{4Ate^{-t}} + \cancel{At^2e^{-t}} + \cancel{4Ate^{-t}} - \cancel{2At^2e^{-t}} + \cancel{At^2e^{-t}} = 2e^{-t}$$

$$2Ae^{-t} = 2e^{-t}$$

For this equation to be true, A must satisfy

$$2A = 2,$$

or $A = 1$.

$$y_p(t) = t^2e^{-t}.$$

Therefore,

$$y(t) = C_1 te^{-t} + C_2 e^{-t} + t^2e^{-t}.$$