Problem 9

In each of Problems 1 through 14, find the general solution of the given differential equation.

\[ 2y'' + 3y' + y = t^2 + 3\sin t \]

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ 2y'' + 3y' + y_c = 0 \] (1)

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \quad \rightarrow \quad y'_c = re^{rt} \quad \rightarrow \quad y''_c = r^2e^{rt} \]

Substitute these expressions into the ODE.

\[ 2(r^2e^{rt}) + 3(re^{rt}) + e^{rt} = 0 \]

Divide both sides by \( e^{rt} \).

\[ 2r^2 + 3r + 1 = 0 \]
\[ (2r + 1)(r + 1) = 0 \]

\[ r = \left\{ -1, -\frac{1}{2} \right\} \]

Two solutions to equation (1) are then \( y_c = e^{-t} \) and \( y_c = e^{-t/2} \). By the principle of superposition, the general solution is a linear combination of these two.

\[ y_c(t) = C_1e^{-t} + C_2e^{-t/2} \]

On the other hand, the particular solution satisfies

\[ 2y''_p + 3y'_p + y_p = t^2 + 3\sin t \]

The inhomogeneous term has two components, a monomial and a sine function. Since both odd and even derivatives are present, both sine and cosine need to be included in the trial solution. For the monomial, all powers of \( t \) leading up to and including \( t^2 \) must be included as well. The trial solution is thus \( y_p(t) = A + Bt + Ct^2 + D\cos t + E\sin t \). Substitute this into the ODE to determine \( A, B, C, D, \) and \( E \).

\[ 2(A + Bt + Ct^2 + D\cos t + E\sin t)'' + 3(A + Bt + Ct^2 + D\cos t + E\sin t)'+(A + Bt + Ct^2 + D\cos t + E\sin t) = t^2 + 3\sin t \]

\[ 2(B + 2Ct - D \sin t + E \cos t)' + 3(B + 2Ct - D \sin t + E \cos t) + (A + Bt + Ct^2 + D \cos t + E \sin t) = t^2 + 3\sin t \]

\[ 2(2C - D \cos t - E \sin t) + 3(B + 2Ct - D \sin t + E \cos t) + (A + Bt + Ct^2 + D \cos t + E \sin t) = t^2 + 3\sin t \]

\[ 4C - 2D \cos t - 2E \sin t + 3B + 6Ct - 3D \sin t + 3E \cos t + A + Bt + Ct^2 + D \cos t + E \sin t = t^2 + 3\sin t \]
For this equation to be true, $A$, $B$, $C$, $D$, and $E$ must satisfy the following system of equations.

\begin{align*}
4C + 3B + A &= 0 \\
6C + B &= 0 \\
C &= 1 \\
-2D + 3E + D &= 0 \\
-2E - 3D + E &= 3
\end{align*}

Solving it yields $A = 14$, $B = -6$, $C = 1$, $D = -9/10$, and $E = -3/10$, which means

\[ y_p(t) = 14 - 6t + t^2 - \frac{9}{10} \cos t - \frac{3}{10} \sin t. \]

Therefore,

\[ y(t) = C_1 e^{-t} + C_2 e^{-t/2} + 14 - 6t + t^2 - \frac{9}{10} \cos t - \frac{3}{10} \sin t. \]