Problem 10

In each of Problems 1 through 14, find the general solution of the given differential equation.

\[ y'' + y = 3 \sin 2t + t \cos 2t \]

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y'' + y = 0 \quad (1) \]

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \rightarrow y'_c = re^{rt} \rightarrow y''_c = r^2e^{rt} \]

Substitute these expressions into the ODE.

\[ r^2e^{rt} + e^{rt} = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 + 1 = 0 \]

\[ r = \{-i, i\} \]

Two solutions to equation (1) are then \( y_c = e^{-it} \) and \( y_c = e^{it} \). By the principle of superposition, the general solution is a linear combination of these two.

\[ y_c(t) = C_1e^{-it} + C_2e^{it} \]

\[ = C_1[\cos(-t) + i\sin(-t)] + C_2[\cos(t) + i\sin(t)] \]

\[ = C_1[\cos(t) - i\sin(t)] + C_2[\cos(t) + i\sin(t)] \]

\[ = C_1 \cos t - iC_1 \sin t + C_2 \cos t + iC_2 \sin t \]

\[ = (C_1 + C_2) \cos t + (-iC_1 + iC_2) \sin t \]

\[ = C_3 \cos t + C_4 \sin t \]

On the other hand, the particular solution satisfies

\[ y''_p + y_p = 3 \sin 2t + t \cos 2t. \]

There are two terms on the right side. For the first one, since only even derivatives are present, we will include \( A \sin 2t \) in the trial solution. For the second one, we will include \( Bt \cos 2t + C \sin 2t \). The trial solution is thus \( y_p(t) = A \sin 2t + Bt \cos 2t + C \sin 2t \). Substitute this into the ODE to determine \( A \) and \( B \) and \( C \).

\[ (A \sin 2t + Bt \cos 2t + C \sin 2t)'' + (A \sin 2t + Bt \cos 2t + C \sin 2t) = 3 \sin 2t + t \cos 2t \]

\[ (2A \cos 2t + B \cos 2t - 2Bt \sin 2t + 2C \cos 2t)' + (A \sin 2t + Bt \cos 2t + C \sin 2t) = 3 \sin 2t + t \cos 2t \]

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\((-4A \sin 2t - 2B \sin 2t - 2B \sin 2t - 4B \cos 2t - 4C \sin 2t) + (A \sin 2t + Bt \cos 2t + C \sin 2t) = 3 \sin 2t + t \cos 2t\)
\((-4A - 2B - 2B - 4C + A + C) \sin 2t + (-4B + B)t \cos 2t = 3 \sin 2t + t \cos 2t\)

For this equation to be true, \(A\) and \(B\) and \(C\) must satisfy the following system of equations.

\[-4A - 2B - 2B - 4C + A + C = 3\]
\[-4B + B = 1\]

Solving it yields \(A + C = -5/9\) and \(B = -1/3\), which means

\[y_p(t) = A \sin 2t + Bt \cos 2t + C \sin 2t\]
\[= (A + C) \sin 2t + Bt \cos 2t\]
\[= \frac{-5}{9} \sin 2t - \frac{1}{3} t \cos 2t.\]

Therefore,

\[y(t) = C_3 \cos t + C_4 \sin t - \frac{5}{9} \sin 2t - \frac{1}{3} t \cos 2t.\]