# Problem 11

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$u'' + \omega_0^2 u = \cos \omega t, \qquad \omega^2 \neq \omega_0^2$$

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### Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $u_c(t)$ and the particular solution $u_p(t)$.

$$u(t) = u_c(t) + u_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$u_c'' + \omega_0^2 u_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $u_c = e^{rt}$.

$$u_c = e^{rt} \quad \rightarrow \quad u_c' = re^{rt} \quad \rightarrow \quad u_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + \omega_0^2 (e^{rt}) = 0$$

Divide both sides by $e^{rt}$.

$$r^2 + \omega_0^2 = 0$$

$$r = \{-i\omega_0, i\omega_0\}$$

Two solutions to equation (1) are then $u_c = e^{-i\omega_0 t}$ and $u_c = e^{i\omega_0 t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
u_c(t) &= C_1 e^{-i\omega_0 t} + C_2 e^{i\omega_0 t} \\
&= C_1[\cos(-\omega_0 t) + i \sin(-\omega_0 t)] + C_2[\cos(\omega_0 t) + i \sin(\omega_0 t)] \\
&= C_1[\cos(\omega_0 t) - i \sin(\omega_0 t)] + C_2[\cos(\omega_0 t) + i \sin(\omega_0 t)] \\
&= C_1 \cos \omega_0 t - iC_1 \sin \omega_0 t + C_2 \cos \omega_0 t + iC_2 \sin \omega_0 t \\
&= (C_1 + C_2) \cos \omega_0 t + (-iC_1 + iC_2) \sin \omega_0 t \\
&= C_3 \cos \omega_0 t + C_4 \sin \omega_0 t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$u_p'' + \omega_0^2 u_p = \cos \omega t.$$

Since only even derivatives are present, the trial solution will be $u_p(t) = A \cos \omega t$. Substitute this into the ODE to determine $A$.

$$(A \cos \omega t)'' + \omega_0^2 (A \cos \omega t) = \cos \omega t$$

$$(-A\omega \sin \omega t)' + \omega_0^2 (A \cos \omega t) = \cos \omega t$$

$$(-A\omega^2 \cos \omega t) + \omega_0^2 (A \cos \omega t) = \cos \omega t$$

$$(-A\omega^2 + A\omega_0^2)\cos\omega t = \cos\omega t$$

For this equation to be true, $A$ must satisfy

$$-A\omega^2 + A\omega_0^2 = 1,$$

or $A = 1/(\omega^2 - \omega_0^2)$.

$$u_p(t) = \frac{1}{\omega^2 - \omega_0^2}\cos\omega t$$

Therefore,

$$u(t) = C_3\cos\omega_0 t + C_4\sin\omega_0 t + \frac{1}{\omega^2 - \omega_0^2}\cos\omega t.$$