Problem 13

In each of Problems 1 through 14, find the general solution of the given differential equation.

\[ y'' + y' + 4y = 2\sinh t \]

*Hint:* \( \sinh t = (e^t - e^{-t})/2 \)

**Solution**

Rewrite the right side in terms of exponential functions.

\[ y'' + y' + 4y = e^t - e^{-t} \]

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y''_c + y'_c + 4y_c = 0 \] (1)

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \rightarrow y'_c = re^{rt} \rightarrow y''_c = r^2e^{rt} \]

Substitute these expressions into the ODE.

\[ r^2e^{rt} + re^{rt} + 4(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 + r + 4 = 0 \]

\[ r = \frac{-1 \pm \sqrt{1 - 4(1)(4)}}{2} = \frac{-1 \pm \sqrt{-15}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{15}}{2} \]

\[ r = \left\{ -\frac{1}{2} - \frac{i\sqrt{15}}{2}, -\frac{1}{2} + \frac{i\sqrt{15}}{2} \right\} \]

Two solutions to equation (1) are then \( y_c = e^{(-1/2-i\sqrt{15}/2)t} \) and \( y_c = e^{(-1/2+i\sqrt{15}/2)t} \).
By the principle of superposition, the general solution is a linear combination of these two.

\[ y_c(t) = C_1 e^{-t/2 - i\sqrt{15}/2} + C_2 e^{-t/2 + i\sqrt{15}/2} \]

\[ = C_1 e^{-t} e^{-i\sqrt{15}t/2} + C_2 e^{-t} e^{i\sqrt{15}t/2} \]

\[ = C_1 e^{-t/2} e^{-i\sqrt{15}t/2} + C_2 e^{-t/2} e^{i\sqrt{15}t/2} \]

\[ = C_1 \left[ \cos \left( \frac{-\sqrt{15}}{2} t \right) + i \sin \left( -\frac{\sqrt{15}}{2} t \right) \right] + C_2 \left[ \cos \left( \frac{\sqrt{15}}{2} t \right) + i \sin \left( \frac{\sqrt{15}}{2} t \right) \right] \]

\[ = C_1 \left[ \cos \left( \frac{\sqrt{15}}{2} t \right) - i \sin \left( \frac{\sqrt{15}}{2} t \right) \right] + C_2 \left[ \cos \left( \frac{\sqrt{15}}{2} t \right) + i \sin \left( \frac{\sqrt{15}}{2} t \right) \right] \]

\[ = C_1 e^{-t/2} \cos \left( \frac{\sqrt{15}}{2} t \right) - iC_1 e^{-t/2} \sin \left( \frac{\sqrt{15}}{2} t \right) + C_2 e^{-t/2} \cos \left( \frac{\sqrt{15}}{2} t \right) + iC_2 e^{-t/2} \sin \left( \frac{\sqrt{15}}{2} t \right) \]

\[ = (C_1 + C_2)e^{-t/2} \cos \left( \frac{\sqrt{15}}{2} t \right) + (-iC_1 + iC_2)e^{-t/2} \sin \left( \frac{\sqrt{15}}{2} t \right) \]

\[ = C_3 e^{-t/2} \cos \left( \frac{\sqrt{15}}{2} t \right) + C_4 e^{-t/2} \sin \left( \frac{\sqrt{15}}{2} t \right) \]

On the other hand, the particular solution satisfies

\[ y''_p + y'_p + 4y_p = e^t - e^{-t}. \]

There are two terms on the right side. For the first one, we will include \( Ae^t \) in the trial solution. For the second one, we will include \( Be^{-t} \). The trial solution is thus \( y_p(t) = Ae^t + Be^{-t} \).

Substitute this into the ODE to determine \( A \) and \( B \).

\[ (Ae^t + Be^{-t})'' + (Ae^t + Be^{-t})' + 4(Ae^t + Be^{-t}) = e^t - e^{-t}. \]

\[ (Ae^t - Be^{-t})' + (Ae^t - Be^{-t}) + 4(Ae^t + Be^{-t}) = e^t - e^{-t}. \]

\[ (Ae^t + Be^{-t}) + (Ae^t - Be^{-t}) + 4(Ae^t + Be^{-t}) = e^t - e^{-t}. \]

\[ (A + A + 4A)e^t + (B - B + 4B)e^{-t} = e^t - e^{-t}. \]

For this equation to be true, \( A \) and \( B \) must satisfy the following system of equations.

\[ A + A + 4A = 1 \]

\[ B - B + 4B = -1 \]

Solving it yields \( A = 1/6 \) and \( B = -1/4 \), which means

\[ y_p(t) = \frac{1}{6} e^t - \frac{1}{4} e^{-t}. \]

Therefore,

\[ y(t) = C_3 e^{-t/2} \cos \left( \frac{\sqrt{15}}{2} t \right) + C_4 e^{-t/2} \sin \left( \frac{\sqrt{15}}{2} t \right) + \frac{1}{6} e^t - \frac{1}{4} e^{-t}. \]