Problem 15

In each of Problems 15 through 20, find the solution of the given initial value problem.

\[ y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1 \]

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y''_c + y'_c - 2y_c = 0 \]  \hspace{1cm} (1)

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \rightarrow y'_c = re^{rt} \rightarrow y''_c = r^2e^{rt} \]

Substitute these expressions into the ODE.

\[ r^2e^{rt} + re^{rt} - 2(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 + r - 2 = 0 \]
\[ (r + 2)(r - 1) = 0 \]
\[ r = \{-2, 1\} \]

Two solutions to equation (1) are then \( y_c = e^{-2t} \) and \( y_c = e^t \). By the principle of superposition, the general solution is a linear combination of these two.

\[ y_c(t) = C_1e^{-2t} + C_2e^t \]

On the other hand, the particular solution satisfies

\[ y''_p + y'_p - 2y_p = 2t. \]

The trial solution we will use is \( y_p(t) = A + Bt \), a polynomial that ends at the highest power of \( t \). Substitute this into the ODE to determine \( A \) and \( B \).

\[ (A + Bt)'' + (A + Bt)' - 2(A + Bt) = 2t \]
\[ (B)'' + (B)' - 2(A + Bt) = 2t \]
\[ (0) + (B) - 2(A + Bt) = 2t \]
\[ (B - 2A) + (-2B)t = 2t \]

For this equation to be true, \( A \) and \( B \) must satisfy the following system of equations.

\[ B - 2A = 0 \]
\[ -2B = 2 \]
Solving it yields $A = -1/2$ and $B = -1$, which means

$$y_p(t) = -\frac{1}{2} - t.$$ 

The general solution is then

$$y(t) = C_1 e^{-2t} + C_2 e^t - \frac{1}{2} - t.$$ 

Take a derivative of it with respect to $t$.

$$y'(t) = -2C_1 e^{-2t} + C_2 e^t - 1$$

Apply the initial conditions now to determine $C_1$ and $C_2$.

$$y(0) = C_1 + C_2 - \frac{1}{2} = 0$$
$$y'(0) = -2C_1 + C_2 - 1 = 1$$

Solving this system of equations yields $C_1 = -1/2$ and $C_2 = 1$. Therefore,

$$y(t) = -\frac{1}{2} e^{-2t} + e^t - \frac{1}{2} - t.$$