Problem 18

In each of Problems 15 through 20, find the solution of the given initial value problem.

$$y'' - 2y' - 3y = 3te^{2t}, \qquad y(0) = 1, \quad y'(0) = 0$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 2y_c' - 3y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 2(re^{rt}) - 3(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 2r - 3 = 0$$
$$(r-3)(r+1) = 0$$
$$r = \{-1, 3\}$$

Two solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^{3t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1 e^{-t} + C_2 e^{3t}$$

On the other hand, the particular solution satisfies

$$y_p'' - 2y_p' - 3y_p = 3te^{2t}.$$

The trial solution we will use is $y_p(t) = (A + Bt)e^{2t}$. Substitute this into the ODE to determine A and B.

$$[(A+Bt)e^{2t}]'' - 2[(A+Bt)e^{2t}]' - 3[(A+Bt)e^{2t}] = 3te^{2t}$$
$$[Be^{2t} + 2(A+Bt)e^{2t}]' - 2[Be^{2t} + 2(A+Bt)e^{2t}] - 3[(A+Bt)e^{2t}] = 3te^{2t}$$
$$[2Be^{2t} + 2Be^{2t} + 4(A+Bt)e^{2t}] - 2[Be^{2t} + 2(A+Bt)e^{2t}] - 3[(A+Bt)e^{2t}] = 3te^{2t}$$
$$2Be^{2t} + \cancel{2Be^{2t}} + \cancel{4(A+Bt)e^{2t}} - \cancel{2Be^{2t}} - \cancel{4(A+Bt)e^{2t}} - 3(A+Bt)e^{2t} = 3te^{2t}$$
$$(2B - 3A)e^{2t} + (-3B)te^{2t} = 3te^{2t}$$

For this equation to be true, A and B must satisfy the following system of equations.

$$2B - 3A = 0$$
$$-3B = 3$$

Solving it yields $A = -2/3$ and $B = -1$, which means

$$y_p(t) = (A + Bt)e^{2t}.$$

The general solution is then

$$y(t) = C_1 e^{-t} + C_2 e^{3t} + \left(-\frac{2}{3} - t\right) e^{2t}.$$

Take a derivative of it with respect to t.

$$y'(t) = -C_1 e^{-t} + 3C_2 e^{3t} - e^{2t} + 2\left(-\frac{2}{3} - t\right) e^{2t}$$

Apply the initial conditions now to determine C_1 and C_2.

$$y(0) = C_1 + C_2 - \frac{2}{3} = 1$$
$$y'(0) = -C_1 + 3C_2 - 1 - \frac{4}{3} = 0$$

Solving this system of equations yields $C_1 = 2/3$ and $C_2 = 1$. Therefore,

$$y(t) = \frac{2}{3} e^{-t} + e^{3t} + \left(-\frac{2}{3} - t\right) e^{2t}.$$