Problem 19

In each of Problems 15 through 20, find the solution of the given initial value problem.

$$y'' + 4y = 3\sin 2t, \qquad y(0) = 2, \quad y'(0) = -1$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 4y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 4 = 0$$

$$r = \{-2i, 2i\}$$

Two solutions to equation (1) are then $y_c = e^{-2it}$ and $y_c = e^{2it}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1 e^{-2it} + C_2 e^{2it} \\
&= C_1[\cos(-2t) + i\sin(-2t)] + C_2[\cos(2t) + i\sin(2t)] \\
&= C_1[\cos(2t) - i\sin(2t)] + C_2[\cos(2t) + i\sin(2t)] \\
&= C_1 \cos 2t - iC_1 \sin 2t + C_2 \cos 2t + iC_2 \sin 2t \\
&= (C_1 + C_2)\cos 2t + (-iC_1 + iC_2)\sin 2t \\
&= C_3 \cos 2t + C_4 \sin 2t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + 4y_p = 3\sin 2t.$$

Since the inhomogeneous term is sine and there are only even derivatives, the trial solution would be $A\sin 2t$. However, $\sin 2t$ is a solution to equation (1). An extra factor of t is needed as a result, but this introduces cosine into the equation. The trial solution we will use then is $y_p(t) = t(A\cos 2t + B\sin 2t)$. Substitute this into the ODE to determine A and B.

$$[t(A\cos 2t + B\sin 2t)]'' + 4[t(A\cos 2t + B\sin 2t)] = 3\sin 2t$$

$$[(A\cos 2t + B\sin 2t) + t(-2A\sin 2t + 2B\cos 2t)]' + 4[t(A\cos 2t + B\sin 2t)] = 3\sin 2t$$

$$[(-2A\sin 2t+2B\cos 2t)+(-2A\sin 2t+2B\cos 2t)+t(-4A\cos 2t-4B\sin 2t)]+4[t(A\cos 2t+B\sin 2t)] = 3\sin 2t$$

$$-2A\sin 2t+2B\cos 2t-2A\sin 2t+2B\cos 2t-\cancel{4At\cos 2t}-\cancel{4Bt\sin 2t}+\cancel{4At\cos 2t}+\cancel{4Bt\sin 2t} = 3\sin 2t$$

$$-4A\sin 2t + 4B\cos 2t = 3\sin 2t$$

For this equation to be true, A and B must satisfy the following system of equations.

$$\begin{aligned} -4A &= 3 \\ 4B &= 0 \end{aligned}$$

Solving it yields $A = -3/4$ and $B = 0$, which means

$$\begin{aligned} y_p(t) &= t\left(-\frac{3}{4}\cos 2t + 0\sin 2t\right) \\ &= -\frac{3}{4}t\cos 2t. \end{aligned}$$

The general solution is then

$$y(t) = C_3\cos 2t + C_4\sin 2t - \frac{3}{4}t\cos 2t.$$

Take a derivative of it with respect to t.

$$y'(t) = -2C_3\sin 2t + 2C_4\cos 2t - \frac{3}{4}\cos 2t + \frac{3}{2}t\sin 2t$$

Apply the initial conditions now to determine C_3 and C_4.

$$\begin{aligned} y(0) &= C_3 = 2 \\ y'(0) &= 2C_4 - \frac{3}{4} = -1 \end{aligned}$$

Solving this system of equations yields $C_3 = 2$ and $C_4 = -1/8$. Therefore,

$$y(t) = 2\cos 2t - \frac{1}{8}\sin 2t - \frac{3}{4}t\cos 2t.$$