Problem 20

In each of Problems 15 through 20, find the solution of the given initial value problem.

\[ y'' + 2y' + 5y = 4e^{-t}\cos 2t, \quad y(0) = 1, \quad y'(0) = 0 \]

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y'' + 2y' + 5y = 0 \]  \hspace{1cm} (1)

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \rightarrow y'_c = re^{rt} \rightarrow y''_c = r^2e^{rt} \]

Substitute these expressions into the ODE.

\[ r^2e^{rt} + 2(re^{rt}) + 5(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 + 2r + 5 = 0 \]

\[ r = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i \]

Two solutions to equation (1) are then \( y_c = e^{(-1-2i)t} \) and \( y_c = e^{(-1+2i)t} \). By the principle of superposition, the general solution is a linear combination of these two.

\[ y_c(t) = C_1e^{(-1-2i)t} + C_2e^{(-1+2i)t} \]

\[ = C_1e^{-t-2it} + C_2e^{-t+2it} \]

\[ = C_1e^{-t}e^{-2it} + C_2e^{-t}e^{2it} \]

\[ = C_1e^{-t}[\cos(-2t) + i\sin(-2t)] + C_2e^{-t}[\cos(2t) + i\sin(2t)] \]

\[ = C_1e^{-t}[\cos(2t) - i\sin(2t)] + C_2e^{-t}[\cos(2t) + i\sin(2t)] \]

\[ = C_1e^{-t}\cos 2t - iC_1e^{-t}\sin 2t + C_2e^{-t}\cos 2t + iC_2e^{-t}\sin 2t \]

\[ = (C_1 + C_2)e^{-t}\cos 2t + (-iC_1 + iC_2)e^{-t}\sin 2t \]

\[ = C_3e^{-t}\cos 2t + C_4e^{-t}\sin 2t \]

On the other hand, the particular solution satisfies

\[ y''_p + 2y'_p + 5y_p = 4e^{-t}\cos 2t. \]

Since both even and odd derivatives are present, we would use the trial solution \( e^{-t}(A\cos 2t + B\sin 2t) \) rather than \( Ae^{-t}\cos 2t \). However, \( e^{-t}\cos 2t \) and \( e^{-t}\sin 2t \) satisfy equation...
The general solution is then

\[ y_p(t) = te^{-t}(A \cos 2t + B \sin 2t). \]

Substitute this into the ODE to determine \( A \) and \( B \).

\[
[te^{-t}(A \cos 2t + B \sin 2t)]'' + 2[te^{-t}(A \cos 2t + B \sin 2t)]' + 5[te^{-t}(A \cos 2t + B \sin 2t)] = 4e^{-t} \cos 2t.
\]

Solving this system of equations yields

\[
\begin{align*}
[&e^{-t}(A \cos 2t + B \sin 2t) - te^{-t}(A \cos 2t + B \sin 2t) + te^{-t}(-2A \sin 2t + 2B \cos 2t) \\
&+ 2[e^{-t}(A \cos 2t + B \sin 2t) - te^{-t}(A \cos 2t + B \sin 2t) + te^{-t}(-2A \sin 2t + 2B \cos 2t)] \\
&+ 5[te^{-t}(A \cos 2t + B \sin 2t)] = 4e^{-t} \cos 2t.
\end{align*}
\]

For this equation to be true, \( A \) and \( B \) must satisfy the following system of equations.

\[
\begin{align*}
-4A &= 0 \\
4B &= 4
\end{align*}
\]

Solving it yields \( A = 0 \) and \( B = 1 \), which means

\[
y_p(t) = te^{-t}(0 \cos 2t + \sin 2t) \\
= te^{-t} \sin 2t.
\]

The general solution is then

\[ y(t) = C_3 e^{-t} \cos 2t + C_4 e^{-t} \sin 2t + te^{-t} \sin 2t. \]

Take a derivative of it with respect to \( t \).

\[
y'(t) = -C_3 e^{-t} \cos 2t - 2C_3 e^{-t} \sin 2t - C_4 e^{-t} \sin 2t + 2C_4 e^{-t} \cos 2t + e^{-t} \sin 2t - te^{-t} \sin 2t + 2te^{-t} \cos 2t
\]

Apply the initial conditions now to determine \( C_3 \) and \( C_4 \).

\[
\begin{align*}
y(0) &= C_3 = 1 \\
y'(0) &= -C_3 + 2C_4 = 0
\end{align*}
\]

Solving this system of equations yields \( C_3 = 1 \) and \( C_4 = 1/2 \). Therefore,

\[ y(t) = e^{-t} \cos 2t + \frac{1}{2} e^{-t} \sin 2t + te^{-t} \sin 2t. \]