Problem 20

In each of Problems 15 through 20, find the solution of the given initial value problem.

$$y'' + 2y' + 5y = 4e^{-t}\cos 2t, \qquad y(0) = 1, \quad y'(0) = 0$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 2y_c' + 5y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + 2(re^{rt}) + 5(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$r = \{-1 - 2i, -1 + 2i\}$$

Two solutions to equation (1) are then $y_c = e^{(-1-2i)t}$ and $y_c = e^{(-1+2i)t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1e^{(-1-2i)t} + C_2e^{(-1+2i)t} \\
&= C_1e^{-t-2it} + C_2e^{-t+2it} \\
&= C_1e^{-t}e^{-2it} + C_2e^{-t}e^{2it} \\
&= C_1e^{-t}[\cos(-2t) + i\sin(-2t)] + C_2e^{-t}[\cos(2t) + i\sin(2t)] \\
&= C_1e^{-t}[\cos(2t) - i\sin(2t)] + C_2e^{-t}[\cos(2t) + i\sin(2t)] \\
&= C_1e^{-t}\cos 2t - iC_1e^{-t}\sin 2t + C_2e^{-t}\cos 2t + iC_2e^{-t}\sin 2t \\
&= (C_1 + C_2)e^{-t}\cos 2t + (-iC_1 + iC_2)e^{-t}\sin 2t \\
&= C_3e^{-t}\cos 2t + C_4e^{-t}\sin 2t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + 2y_p' + 5y_p = 4e^{-t}\cos 2t.$$

Since both even and odd derivatives are present, we would use the trial solution $e^{-t}(A\cos 2t + B\sin 2t)$ rather than $Ae^{-t}\cos 2t$. However, $e^{-t}\cos 2t$ and $e^{-t}\sin 2t$ satisfy equation

(1), so an extra factor of t is needed. The trial solution is thus $y_p(t) = te^{-t}(A\cos 2t + B\sin 2t)$. Substitute this into the ODE to determine A and B.

$$[te^{-t}(A\cos 2t + B\sin 2t)]'' + 2[te^{-t}(A\cos 2t + B\sin 2t)]' + 5[te^{-t}(A\cos 2t + B\sin 2t)] = 4e^{-t}\cos 2t.$$

$$\begin{aligned}[e^{-t}(A\cos 2t + B\sin 2t) - te^{-t}(A\cos 2t + B\sin 2t) + te^{-t}(-2A\sin 2t + 2B\cos 2t)]' \\ + 2[e^{-t}(A\cos 2t + B\sin 2t) - te^{-t}(A\cos 2t + B\sin 2t) + te^{-t}(-2A\sin 2t + 2B\cos 2t)] \\ + 5[te^{-t}(A\cos 2t + B\sin 2t)] = 4e^{-t}\cos 2t.\end{aligned}$$

$$\begin{aligned}[-e^{-t}(A\cos 2t+B\sin 2t)+e^{-t}(-2A\sin 2t+2B\cos 2t)-e^{-t}(A\cos 2t+B\sin 2t)+te^{-t}(A\cos 2t+B\sin 2t) \\ - te^{-t}(-2A\sin 2t + 2B\cos 2t) + e^{-t}(-2A\sin 2t + 2B\cos 2t) \\ - te^{-t}(-2A\sin 2t + 2B\cos 2t) + te^{-t}(-4A\cos 2t - 4B\sin 2t)] \\ + 2[e^{-t}(A\cos 2t + B\sin 2t) - te^{-t}(A\cos 2t + B\sin 2t) + te^{-t}(-2A\sin 2t + 2B\cos 2t)] \\ + 5[te^{-t}(A\cos 2t + B\sin 2t)] = 4e^{-t}\cos 2t.\end{aligned}$$

$$-4Ae^{-t}\sin 2t + 4Be^{-t}\cos 2t = 4e^{-t}\cos 2t$$

For this equation to be true, A and B must satisfy the following system of equations.

$$\begin{aligned}-4A &= 0 \\ 4B &= 4\end{aligned}$$

Solving it yields $A = 0$ and $B = 1$, which means

$$\begin{aligned}y_p(t) &= te^{-t}(0\cos 2t + \sin 2t) \\ &= te^{-t}\sin 2t.\end{aligned}$$

The general solution is then

$$y(t) = C_3e^{-t}\cos 2t + C_4e^{-t}\sin 2t + te^{-t}\sin 2t.$$

Take a derivative of it with respect to t.

$$y'(t) = -C_3e^{-t}\cos 2t - 2C_3e^{-t}\sin 2t - C_4e^{-t}\sin 2t + 2C_4e^{-t}\cos 2t + e^{-t}\sin 2t - te^{-t}\sin 2t + 2te^{-t}\cos 2t$$

Apply the initial conditions now to determine C_3 and C_4.

$$\begin{aligned}y(0) &= C_3 = 1 \\ y'(0) &= -C_3 + 2C_4 = 0\end{aligned}$$

Solving this system of equations yields $C_3 = 1$ and $C_4 = 1/2$. Therefore,

$$y(t) = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t + te^{-t}\sin 2t.$$