Problem 21

In each of Problems 21 through 28:

(a) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

$$y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin 3t$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 3y_c' = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + 3(re^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 3r = 0$$

$$r(r+3) = 0$$

$$r = \{-3, 0\}$$

Two solutions to equation (1) are then $y_c = e^{-3t}$ and $y_c = e^0 = 1$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1e^{-3t} + C_2$$

On the other hand, the particular solution satisfies

$$y_p'' + 3y_p' = 2t^4 + t^2e^{-3t} + \sin 3t.$$

There are three terms on the right side. For the first one, we would include $A + Bt + Ct^2 + Dt^3 + Et^4$ in the trial solution, but because a constant satisfies equation (1), an extra factor of t is needed. For the second one, we would include $(F + Gt + Ht^2)e^{-3t}$ in the trial solution, but because e^{-3t} satisfies equation (1), an extra factor of t is needed. For the third one, we will include $I\cos 3t + J\sin 3t$ in the trial solution since both even and odd derivatives are present. The trial solution is thus

$$y_p(t) = t(A + Bt + Ct^2 + Dt^3 + Et^4) + t(F + Gt + Ht^2)e^{-3t} + I\cos 3t + J\sin 3t.$$

Substitute this into the ODE to determine A, B, C, D, E, F, G, H, I, and J.

$$\begin{aligned}[t(A + Bt + Ct^2 + Dt^3 + Et^4) + t(F + Gt + Ht^2)e^{-3t} + I\cos 3t + J\sin 3t]'' \\ + 3[t(A + Bt + Ct^2 + Dt^3 + Et^4) + t(F + Gt + Ht^2)e^{-3t} + I\cos 3t + J\sin 3t]' \\ = 2t^4 + t^2e^{-3t} + \sin 3t\end{aligned}$$

Evaluate the derivatives and fully simplify the left side.

$$\begin{aligned}(3A + 2B) + (6B + 6C)t + (9C + 12D)t^2 + (12D + 20E)t^3 + (15E)t^4 \\ + (-3F + 2G)e^{-3t} + (-6G + 6H)te^{-3t} + (-9H)t^2e^{-3t} + (-9I + 9J)\cos 3t \\ + (-9I - 9J)\sin 3t = 2t^4 + t^2e^{-3t} + \sin 3t\end{aligned}$$

For this equation to be true, A, B, C, D, E, F, G, H, I, and J must satisfy the following system of equations.

$$\begin{aligned}3A + 2B &= 0 \\ 6B + 6C &= 0 \\ 9C + 12D &= 0 \\ 12D + 20E &= 0 \\ 15E &= 2 \\ -3F + 2G &= 0 \\ -6G + 6H &= 0 \\ -9H &= 1 \\ -9I + 9J &= 0 \\ -9I - 9J &= 1\end{aligned}$$

Solving it yields $A = 16/81$, $B = -8/27$, $C = 8/27$, $D = -2/9$, $E = 2/15$, $F = -2/27$, $G = -1/9$, $H = -1/9$, $I = -1/18$, and $J = -1/18$, which means

$$y_p(t) = t\left(\frac{16}{81} - \frac{8}{27}t + \frac{8}{27}t^2 - \frac{2}{9}t^3 + \frac{2}{15}t^4\right) + t\left(-\frac{2}{27} - \frac{1}{9}t - \frac{1}{9}t^2\right)e^{-3t} - \frac{1}{18}\cos 3t - \frac{1}{18}\sin 3t.$$

Therefore,

$$\begin{aligned}y(t) = C_1e^{-3t} + C_2 + t\left(\frac{16}{81} - \frac{8}{27}t + \frac{8}{27}t^2 - \frac{2}{9}t^3 + \frac{2}{15}t^4\right) \\ + t\left(-\frac{2}{27} - \frac{1}{9}t - \frac{1}{9}t^2\right)e^{-3t} - \frac{1}{18}\cos 3t - \frac{1}{18}\sin 3t.\end{aligned}$$