Problem 22

In each of Problems 21 through 28:

(a) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

$$y'' + y = t(1 + \sin t)$$

Solution

Distribute t on the right side.

$$y'' + y = t + t\sin t$$

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 1 = 0$$

$$r = \{-i, i\}$$

Two solutions to equation (1) are then $y_c = e^{-it}$ and $y_c = e^{it}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} y_c(t) &= C_1 e^{-it} + C_2 e^{it} \\ &= C_1[\cos(-t) + i\sin(-t)] + C_2[\cos(t) + i\sin(t)] \\ &= C_1[\cos(t) - i\sin(t)] + C_2[\cos(t) + i\sin(t)] \\ &= C_1\cos t - iC_1\sin t + C_2\cos t + iC_2\sin t \\ &= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t \\ &= C_3\cos t + C_4\sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + y_p = t + t\sin t.$$

There are two terms on the right side. For the first one, we will include $A + Bt$ in the trial solution. For the second one, we would include $(C + Dt)\sin t$ in the trial solution, but this results in cosine terms. An additional $(E+Ft)\cos t$ is needed then to account for this. Also, both $\sin t$ and $\cos t$ satisfy equation (1), so an extra factor of t is needed in each of these. The trial solution is thus

$$y_p(t) = A + Bt + t(C+Dt)\sin t + t(E+Ft)\cos t.$$

Substitute this into the ODE to determine A, B, C, D, E, and F.

$$[A+Bt+t(C+Dt)\sin t + t(E+Ft)\cos t]'' + [A+Bt+t(C+Dt)\sin t + t(E+Ft)\cos t] = t + t\sin t$$

Evaluate the derivatives and fully simplify the left side.

$$A + Bt + (2C+2F)\cos t + (4D)t\cos t + (2D-2E)\sin t + (-4F)t\sin t = t + t\sin t$$

For this equation to be true, A, B, C, D, E, and F must satisfy the following system of equations.

$$\begin{aligned} A &= 0 \\ B &= 1 \\ 2C + 2F &= 0 \\ 4D &= 0 \\ 2D - 2E &= 0 \\ -4F &= 1 \end{aligned}$$

Solving it yields $A = 0$, $B = 1$, $C = 1/4$, $D = 0$, $E = 0$, and $F = -1/4$, which means

$$\begin{aligned} y_p(t) &= t + t\left(\frac{1}{4}\right)\sin t + t\left(-\frac{1}{4}t\right)\cos t \\ &= t + \frac{1}{4}t\sin t - \frac{1}{4}t^2\cos t. \end{aligned}$$

Therefore,

$$y(t) = C_3\cos t + C_4\sin t + t + \frac{1}{4}t\sin t - \frac{1}{4}t^2\cos t.$$