Problem 23

In each of Problems 21 through 28:

(a) Determine a suitable form for \( Y(t) \) if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

\[
y'' - 5y' + 6y = e^t \cos 2t + e^{2t}(3t + 4) \sin t
\]

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[
y(t) = y_c(t) + y_p(t)
\]

The complementary solution satisfies the associated homogeneous equation.

\[
y'' - 5y' + 6y = 0 \quad (1)
\]

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[
y_c = e^{rt} \quad \Rightarrow \quad y'_c = re^{rt} \quad \Rightarrow \quad y''_c = r^2 e^{rt}
\]

Substitute these expressions into the ODE.

\[
r^2 e^{rt} - 5(re^{rt}) + 6(e^{rt}) = 0
\]

Divide both sides by \( e^{rt} \).

\[
r^2 - 5r + 6 = 0
\]

\[
(r - 2)(r - 3) = 0
\]

\[
r = \{2, 3\}
\]

Two solutions to equation (1) are then \( y_c = e^{2t} \) and \( y_c = e^{3t} \). By the principle of superposition, the general solution is a linear combination of these two.

\[
y_c(t) = C_1 e^{2t} + C_2 e^{3t}
\]

On the other hand, the particular solution satisfies

\[
y''_p - 5y'_p + 6y_p = e^t \cos 2t + e^{2t}(3t + 4) \sin t.
\]

There are two terms on the right side. For the first one, we will include \( e^t(A \cos 2t + B \sin 2t) \) in the trial solution. For the second one, we will include \( e^{2t}(C + Dt)(E \cos t + F \sin t) \) in the trial solution. The trial solution is thus

\[
y_p(t) = e^t(A \cos 2t + B \sin 2t) + e^{2t}(C + Dt)(E \cos t + F \sin t).
\]
Substitute this into the ODE to determine $A$, $B$, $C$, $D$, $E$, and $F$.

\[
[e^t(A \cos 2t + B \sin 2t) + e^{2t}(C + Dt)(E \cos t + F \sin t)]'' - 5[e^t(A \cos 2t + B \sin 2t) + e^{2t}(C + Dt)(E \cos t + F \sin t)]' \\
+ 6[e^t(A \cos 2t + B \sin 2t) + e^{2t}(C + Dt)(E \cos t + F \sin t)] = e^t \cos 2t + e^{2t}(3t + 4) \sin t
\]

Evaluate the derivatives and fully simplify the left side. Also, expand the right side.

\[
(-2A - 6B)e^t \cos 2t + (CE - 2DE - CF - DF)e^{2t} \sin t \\
+ (DE - DF)te^{2t} \sin t + (-CE - DE - CF + 2DF)e^{2t} \cos t \\
+ (-DE - DF)te^{2t} \cos t + (6A - 2B)e^t \sin 2t = e^t \cos 2t + 3te^{2t} \sin t + 4e^{2t} \sin t
\]

For this equation to be true, $A$, $B$, $C$, $D$, $E$, and $F$ must satisfy the following system of equations.

\[
-2A - 6B = 1 \\
CE - 2DE - CF - DF = 4 \\
DE - DF = 3 \\
-CE - DE - CF + 2DF = 0 \\
-DE - DF = 0 \\
6A - 2B = 0
\]

Solving it yields $A = -1/20$, $B = -3/20$, $CE = 1/2$, $DE = 3/2$, $CF = -5$, and $DF = -3/2$, which means

\[
y_p(t) = e^t \left(-\frac{1}{20} \cos 2t - \frac{3}{20} \sin 2t\right) + e^{2t}(C + Dt)(E \cos t + F \sin t) \\
- \frac{1}{20}e^t(\cos 2t + 3 \sin 2t) + e^{2t}(CE \cos t + CF \sin t) + te^{2t}(DE \cos t + DF \sin t) \\
- \frac{1}{20}e^t(\cos 2t + 3 \sin 2t) + \frac{1}{2}e^{2t}(\frac{1}{2} \cos t - 5 \sin t) + te^{2t}\left(\frac{3}{2} \cos t - \frac{3}{2} \sin t\right) \\
- \frac{1}{20}e^t(\cos 2t + 3 \sin 2t) + \frac{1}{2}e^{2t}(\cos t - 10 \sin t) + \frac{3}{2}te^{2t}(\cos t - \sin t).
\]

Therefore,

\[
y(t) = C_1e^{2t} + C_2e^{3t} - \frac{1}{20}e^t(\cos 2t + 3 \sin 2t) + \frac{1}{2}e^{2t}(\cos t - 10 \sin t) + \frac{3}{2}te^{2t}(\cos t - \sin t).
\]