Problem 26

In each of Problems 21 through 28:

(a) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

$$y'' + 4y = t^2 \sin 2t + (6t + 7)\cos 2t$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 4y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 4 = 0$$

$$r = \{-2i, 2i\}$$

Two solutions to equation (1) are then $y_c = e^{-2it}$ and $y_c = e^{2it}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1 e^{-2it} + C_2 e^{2it} \\
&= C_1[\cos(-2t) + i\sin(-2t)] + C_2[\cos(2t) + i\sin(2t)] \\
&= C_1[\cos(2t) - i\sin(2t)] + C_2[\cos(2t) + i\sin(2t)] \\
&= C_1 \cos 2t - iC_1 \sin 2t + C_2 \cos 2t + iC_2 \sin 2t \\
&= (C_1 + C_2)\cos 2t + (-iC_1 + iC_2)\sin 2t \\
&= C_3 \cos 2t + C_4 \sin 2t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + 4y_p = t^2 \sin 2t + (6t + 7)\cos 2t.$$

There are two terms on the right side. Both of them can be accounted for by including $(A + Bt + Ct^2)(D\cos 2t + E\sin 2t)$ in the trial solution. However, an extra factor of t is needed because $\cos 2t$ and $\sin 2t$ are solutions to equation (1). The trial solution is thus

$$y_p(t) = t(A + Bt + Ct^2)(D\cos 2t + E\sin 2t).$$

Substitute this into the ODE to determine A, B, C, D, and E.

$$[t(A+Bt+Ct^2)(D\cos 2t+E\sin 2t)]''+4[t(A+Bt+Ct^2)(D\cos 2t+E\sin 2t)] \\ = t^2\sin 2t+(6t+7)\cos 2t$$

Evaluate the derivatives and fully simplify the left side. Also, expand the right side.

$$(2BD+4AE)\cos 2t+(6CD+8BE)t\cos 2t+(12CE)t^2\cos 2t \\ +(-4AD+2BE)\sin 2t+(-8BD+6CE)t\sin 2t+(-12CD)t^2\sin 2t \\ = t^2\sin 2t+7\cos 2t+6t\cos 2t$$

For this equation to be true, A, B, C, D, and E must satisfy the following system of equations.

$$\begin{aligned} 2BD+4AE&=7 \\ 6CD+8BE&=6 \\ 12CE&=0 \\ -4AD+2BE&=0 \\ -8BD+6CE&=0 \\ -12CD&=1 \end{aligned}$$

Solving it yields $BD=0$, $AE=7/4$, $CD=-1/12$, $BE=13/16$, $CE=0$, and $AD=13/32$, which means

$$\begin{aligned} y_p(t)&=t(A+Bt+Ct^2)(D\cos 2t+E\sin 2t) \\ &=ADt\cos 2t+AEt\sin 2t+BDt^2\cos 2t+BEt^2\sin 2t+CDt^3\cos 2t+CEt^3\sin 2t \\ &=\frac{13}{32}t\cos 2t+\frac{7}{4}t\sin 2t+\frac{13}{16}t^2\sin 2t-\frac{1}{12}t^3\cos 2t. \end{aligned}$$

Therefore,

$$y(t)=C_3\cos 2t+C_4\sin 2t+\frac{13}{32}t\cos 2t+\frac{7}{4}t\sin 2t+\frac{13}{16}t^2\sin 2t-\frac{1}{12}t^3\cos 2t.$$