Problem 27

In each of Problems 21 through 28:

(a) Determine a suitable form for \( Y(t) \) if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

\[ y'' + 3y' + 2y = e^t(t^2 + 1) \sin 2t + 3e^{-t} \cos t + 4e^t \]

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y'' + 3y' + 2y = 0 \]

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \quad \Rightarrow \quad y'_c = re^{rt} \quad \Rightarrow \quad y''_c = r^2 e^{rt} \]

Substitute these expressions into the ODE.

\[ r^2 e^{rt} + 3(re^{rt}) + 2(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 + 3r + 2 = 0 \]

\[ (r + 2)(r + 1) = 0 \]

\[ r = \{-2, -1\} \]

Two solutions to equation (1) are then \( y_c = e^{-2t} \) and \( y_c = e^{-t} \). By the principle of superposition, the general solution is a linear combination of these two.

\[ y_c(t) = C_1 e^{-2t} + C_2 e^{-t} \]

On the other hand, the particular solution satisfies

\[ y''_p + 3y'_p + 2y_p = e^t(t^2 + 1) \sin 2t + 3e^{-t} \cos t + 4e^t \]

There are three terms on the right side. For the first one, we will include \( e^t(A + Bt + Ct^2)(D \cos 2t + E \sin 2t) \) in the trial solution. For the second one, we will include \( e^{-t}(F \cos t + G \sin t) \). For the third one, we will include \( He^t \). The trial solution is thus

\[ y_p(t) = e^t(A + Bt + Ct^2)(D \cos 2t + E \sin 2t) + e^{-t}(F \cos t + G \sin t) + He^t. \]
Substitute this into the ODE to determine $A$, $B$, $C$, $D$, $E$, $F$, $G$, and $H$.

\[
[e'(A + Bt + Ct^2)(D \cos 2t + E \sin 2t) + e^{-t}(F \cos t + G \sin t) + He^t]''
+ 3[e'(A + Bt + Ct^2)(D \cos 2t + E \sin 2t) + e^{-t}(F \cos t + G \sin t) + He^t]'
+ 2[e'(A + Bt + Ct^2)(D \cos 2t + E \sin 2t) + e^{-t}(F \cos t + G \sin t) + He^t]
= e'(t^2 + 1) \sin 2t + 3e^{-t} \cos t + 4e^t
\]

Evaluate the derivatives and fully simplify the left side. Also, expand the right side.

\[
(-F + G)e^{-t} \cos t + (-F - G)e^{-t} \sin t + (6H)e^t + (2AD + 5BD + 2CD + 10AE + 4BE)e^t \cos 2t
+ (2BD + 10CD + 10BE + 8CE)te^t \cos 2t + (2CD + 10CE)t^2 \cos 2t
+ (-10AD - 4BD + 2AE + 5BE + 2CE)e^t \sin 2t
+ (-10BD + 8CD + 2BE + 10CE)te^t \sin 2t
+ (-10CD + 2CE)t^2 e^t \sin 2t + t^2 e^t \sin 2t + e^t \sin 2t + 3e^{-t} \cos t + 4e^t
\]

For this equation to be true, $A$, $B$, $C$, $D$, $E$, $F$, $G$, and $H$ must satisfy the following system of equations.

\[
-F + G = 3
-F - G = 0
6H = 4
2AD + 5BD + 2CD + 10AE + 4BE = 0
2BD + 10CD + 10BE + 8CE = 0
2CD + 10CE = 0
-10AD - 4BD + 2AE + 5BE + 2CE = 1
-10BD - 8CD + 2BE + 10CE = 0
-10CD + 2CE = 1
\]

Solving it yields $AD = -4105/35152$, $BD = 73/676$, $CD = -5/52$, $AE = -1233/35152$, $BE = 10/169$, $CE = 1/52$, $F = -3/2$, $G = 3/2$, and $H = 2/3$, which means

\[
y_p(t) = e^t(A + Bt + Ct^2)(D \cos 2t + E \sin 2t) + e^{-t}(F \cos t + G \sin t) + He^t
= AD e^t \cos 2t + AEm e^t \sin 2t + BDte^t \cos 2t + BEm e^t \sin 2t + CD t^2 e^t \cos 2t
+ CEm e^t \sin 2t + F e^{-t} \cos t + Ge^{-t} \sin t + He^t
= -\frac{4105}{35152} e^t \cos 2t - \frac{1233}{35152} e^t \sin 2t + \frac{73}{676} te^t \cos 2t + \frac{10}{169} te^t \sin 2t
- \frac{5}{52} t^2 e^t \cos 2t + \frac{1}{52} t^2 e^t \sin 2t - \frac{3}{2} e^{-t} \cos t + \frac{3}{2} e^{-t} \sin t + \frac{2}{3} e^t.
\]

Therefore,

\[
y(t) = C_1 e^{-2t} + C_2 e^{-t} - \frac{4105}{35152} e^t \cos 2t - \frac{1233}{35152} e^t \sin 2t
+ \frac{73}{676} te^t \cos 2t + \frac{10}{169} te^t \sin 2t - \frac{5}{52} t^2 e^t \cos 2t + \frac{1}{52} t^2 e^t \sin 2t
- \frac{3}{2} e^{-t} \cos t + \frac{3}{2} e^{-t} \sin t + \frac{2}{3} e^t.
\]