Problem 29

Consider the equation

$$y'' - 3y' - 4y = 2e^{-t} \tag{i}$$

from Example 5. Recall that $y_1(t) = e^{-t}$ and $y_2(t) = e^{4t}$ are solutions of the corresponding homogeneous equation. Adapting the method of reduction of order (Section 3.4), seek a solution of the nonhomogeneous equation of the form $Y(t) = v(t)y_1(t) = v(t)e^{-t}$, where $v(t)$ is to be determined.

(a) Substitute $Y(t)$, $Y'(t)$, and $Y''(t)$ into Eq. (i) and show that $v(t)$ must satisfy $v'' - 5v' = 2$.

(b) Let $w(t) = v'(t)$ and show that $w(t)$ must satisfy $w' - 5w = 2$. Solve this equation for $w(t)$.

(c) Integrate $w(t)$ to find $v(t)$ and then show that

$$Y(t) = -\frac{2}{5}te^{-t} + \frac{1}{5}c_1e^{4t} + c_2e^{-t}.$$

The first term on the right side is the desired particular solution of the nonhomogeneous equation. Note that it is a product of t and e^{-t}.

Solution

According to the method of reduction of order, the general solution to Eq. (i) is found by plugging either $v(t)y_1(t)$ or $v(t)y_2(t)$ into it. Here we will use $y(t) = v(t)y_1(t)$.

$$[v(t)e^{-t}]'' - 3[v(t)e^{-t}]' - 4[v(t)e^{-t}] = 2e^{-t}$$

Evaluate the derivatives by using the product rule.

$$[v'(t)e^{-t} - v(t)e^{-t}]' - 3[v'(t)e^{-t} - v(t)e^{-t}] - 4[v(t)e^{-t}] = 2e^{-t}$$

$$[v''(t)e^{-t} - v'(t)e^{-t} - v'(t)e^{-t} + v(t)e^{-t}] - 3[v'(t)e^{-t} - v(t)e^{-t}] - 4[v(t)e^{-t}] = 2e^{-t}$$

$$v''(t)e^{-t} - v'(t)e^{-t} - v'(t)e^{-t} + \cancel{v(t)e^{-t}} - 3v'(t)e^{-t} + \cancel{3v(t)e^{-t}} - \cancel{4v(t)e^{-t}} = 2e^{-t}$$

$$v''(t)e^{-t} - 5v'(t)e^{-t} = 2e^{-t}$$

Multiply both sides by e^t.

$$v''(t) - 5v'(t) = 2$$

This is a first-order linear equation for $v'(t)$, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left[\int^t (-5)\,ds\right] = e^{-5t}$$

Proceed with the multiplication.

$$e^{-5t}v''(t) - 5e^{-5t}v'(t) = 2e^{-5t}$$

The left side can be written as $d/dt(Iv')$ by the chain rule.

$$\frac{d}{dt}(e^{-5t}v') = 2e^{-5t}$$

Integrate both sides with respect to t.

$$e^{-5t}v' = -\frac{2}{5}e^{-5t} + C_1$$

Multiply both sides by e^{5t}.

$$v' = -\frac{2}{5} + C_1e^{5t}$$

Integrate both sides with respect to t once more.

$$v(t) = -\frac{2}{5}t + \frac{C_1}{5}e^{5t} + C_2$$

Therefore, since $y(t) = v(t)e^{-t}$,

$$y(t) = -\frac{2}{5}te^{-t} + \frac{C_1}{5}e^{4t} + C_2e^{-t}.$$