Problem 30

Determine the general solution of

\[ y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin m\pi t. \]

where \( \lambda > 0 \) and \( \lambda \neq m\pi \) for \( m = 1, \ldots, N \).

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y''_c + \lambda^2 y_c = 0 \tag{1} \]

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \rightarrow y'_c = re^{rt} \rightarrow y''_c = r^2e^{rt} \]

Substitute these expressions into the ODE.

\[ r^2e^{rt} + \lambda^2(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 + \lambda^2 = 0 \]

\[ r = \{-i\lambda, i\lambda\} \]

Two solutions to equation (1) are then \( y_c = e^{-i\lambda t} \) and \( y_c = e^{i\lambda t} \). By the principle of superposition, the general solution is a linear combination of these two.

\[ y_c(t) = C_1 e^{-i\lambda t} + C_2 e^{i\lambda t} \]

\[ = C_1 [\cos(-\lambda t) + i \sin(-\lambda t)] + C_2 [\cos(\lambda t) + i \sin(\lambda t)] \]

\[ = C_1 [\cos(\lambda t) - i \sin(\lambda t)] + C_2 [\cos(\lambda t) + i \sin(\lambda t)] \]

\[ = C_1 \cos \lambda t - i C_1 \sin \lambda t + C_2 \cos \lambda t + i C_2 \sin \lambda t \]

\[ = (C_1 + C_2) \cos \lambda t + (-iC_1 + iC_2) \sin \lambda t \]

\[ = C_3 \cos \lambda t + C_4 \sin \lambda t \]

On the other hand, the particular solution satisfies

\[ y''_p + \lambda^2 y_p = \sum_{m=1}^{N} a_m \sin m\pi t. \]

There are \( N \) terms on the right side. Since only even derivatives are present, we can include \( A_1 \sin \pi t \) in the trial solution for the first term, \( A_2 \sin 2\pi t \) for the second term, and so on. The trial solution is thus

\[ y_p(t) = \sum_{m=1}^{N} A_m \sin m\pi t. \]
Substitute this into the ODE to determine $A_1, A_2, \ldots, A_N$.

\[
\left( \sum_{m=1}^{N} A_m \sin m\pi t \right)'' + \lambda^2 \left( \sum_{m=1}^{N} A_m \sin m\pi t \right) = \sum_{m=1}^{N} a_m \sin m\pi t
\]

The derivatives can be brought into the summand because the sum is only finite.

\[
\sum_{m=1}^{N} (A_m \sin m\pi t)'' + \lambda^2 \sum_{m=1}^{N} A_m \sin m\pi t = \sum_{m=1}^{N} a_m \sin m\pi t
\]

Now combine the two sums on the left side.

\[
\sum_{m=1}^{N} (-m^2 \pi^2 A_m \sin m\pi t) + \lambda^2 \sum_{m=1}^{N} A_m \sin m\pi t = \sum_{m=1}^{N} a_m \sin m\pi t
\]

The coefficients must be equal.

\[-m^2 \pi^2 A_m + \lambda^2 A_m = a_m\]

Solve for $A_m$.

\[A_m = \frac{a_m}{\lambda^2 - m^2 \pi^2}\]

The particular solution is then

\[y_p(t) = \sum_{m=1}^{N} A_m \sin m\pi t = \sum_{m=1}^{N} \frac{a_m}{\lambda^2 - m^2 \pi^2} \sin m\pi t.\]

Therefore,

\[y(t) = C_3 \cos \lambda t + C_4 \sin \lambda t + \sum_{m=1}^{N} \frac{a_m}{\lambda^2 - m^2 \pi^2} \sin m\pi t.\]