Problem 31

In many physical problems the nonhomogeneous term may be specified by different formulas in different time periods. As an example, determine the solution $y = \phi(t)$ of

$$y'' + y = \begin{cases} t, & 0 \le t \le \pi, \\ \pi e^{\pi - t}, & t > \pi, \end{cases}$$

satisfying the initial conditions $y(0) = 0$ and $y'(0) = 1$. Assume that y and y' are also continuous at $t = \pi$. Plot the nonhomogeneous term and the solution as functions of time. *Hint:* First solve the initial value problem for $t \le \pi$; then solve for $t > \pi$, determining the constants in the latter solution from the continuity conditions at $t = \pi$.

Solution

Solve the ODE on each time interval separately.

$$y'' + y = t, \quad 0 \le t \le \pi \qquad y'' + y = \pi e^{\pi - t}, \quad t > \pi$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation. It is the same for both intervals.

$$y_c'' + y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 1 = 0$$

$$r = \{-i, i\}$$

Two solutions to equation (1) are then $y_c = e^{-it}$ and $y_c = e^{it}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} y_c(t) &= C_1 e^{-it} + C_2 e^{it} \\ &= C_1[\cos(-t) + i\sin(-t)] + C_2[\cos(t) + i\sin(t)] \\ &= C_1[\cos(t) - i\sin(t)] + C_2[\cos(t) + i\sin(t)] \\ &= C_1 \cos t - iC_1 \sin t + C_2 \cos t + iC_2 \sin t \\ &= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t \end{aligned}$$

Using new constants for the terms in parentheses, the complementary solution is

$$y_c(t) = \begin{cases} C_3 \cos t + C_4 \sin t & \text{if } 0 \le t \le \pi \\ C_5 \cos t + C_6 \sin t & \text{if } t > \pi \end{cases}.$$

On the other hand, the particular solution satisfies

$$y_p'' + y_p = t, \quad 0 \le t \le \pi \qquad y_p'' + y_p = \pi e^{\pi - t}, \quad t > \pi.$$

Since the inhomogeneous term for the first ODE is a monomial, the trial solution is a polynomial of terms leading up to and including the highest order: $y_p(t) = A + Bt$. Since the inhomogeneous term for the second ODE can be written as $\pi e^{\pi} e^{-t}$, the trial solution is Ce^{-t}. Substitute these into the ODEs to determine A, B, and C.

$$(A + Bt)'' + (A + Bt) = t, \quad 0 \le t \le \pi \qquad (Ce^{-t})'' + (Ce^{-t}) = \pi e^{\pi - t}, \quad t > \pi$$

Simplify the left sides.

$$A + Bt = t, \quad 0 \le t \le \pi \qquad 2Ce^{-t} = \pi e^{\pi} e^{-t}, \quad t > \pi$$

For these equations to be true, the following system of equations must be satisfied.

$$A = 0$$
$$B = 1$$
$$2C = \pi e^{\pi}$$

Solving it yields $A = 0$, $B = 1$, and $C = \pi e^{\pi}/2$, which means

$$y_p(t) = \begin{cases} t & \text{if } 0 \le t \le \pi \\ \frac{\pi e^{\pi}}{2} e^{-t} & \text{if } t > \pi \end{cases}.$$

The general solution is then

$$y(t) = \begin{cases} C_3 \cos t + C_4 \sin t + t & \text{if } 0 \le t \le \pi \\ C_5 \cos t + C_6 \sin t + \frac{\pi e^{\pi}}{2} e^{-t} & \text{if } t > \pi \end{cases}.$$

Differentiate it once with respect to t.

$$y'(t) = \begin{cases} -C_3 \sin t + C_4 \cos t + 1 & \text{if } 0 \le t \le \pi \\ -C_5 \sin t + C_6 \cos t - \frac{\pi e^{\pi}}{2} e^{-t} & \text{if } t > \pi \end{cases}$$

The initial conditions are prescribed at $t = 0$, so they can be used to determine C_3 and C_4.

$$y(0) = C_3 = 0$$
$$y'(0) = C_4 + 1 = 1$$

Solving this system of equations yields $C_3 = 0$ and $C_4 = 0$.

$$y(t) = \begin{cases} t & \text{if } 0 \le t \le \pi \\ C_5 \cos t + C_6 \sin t + \frac{\pi e^{\pi}}{2} e^{-t} & \text{if } t > \pi \end{cases}$$

Use the fact that y and y' are continuous at $t = \pi$ to determine C_5 and C_6.

$$\begin{cases} \lim_{t\to\pi^-} y(t) = \lim_{t\to\pi^+} y(t) \\ \lim_{t\to\pi^-} y'(t) = \lim_{t\to\pi^+} y'(t) \end{cases}$$

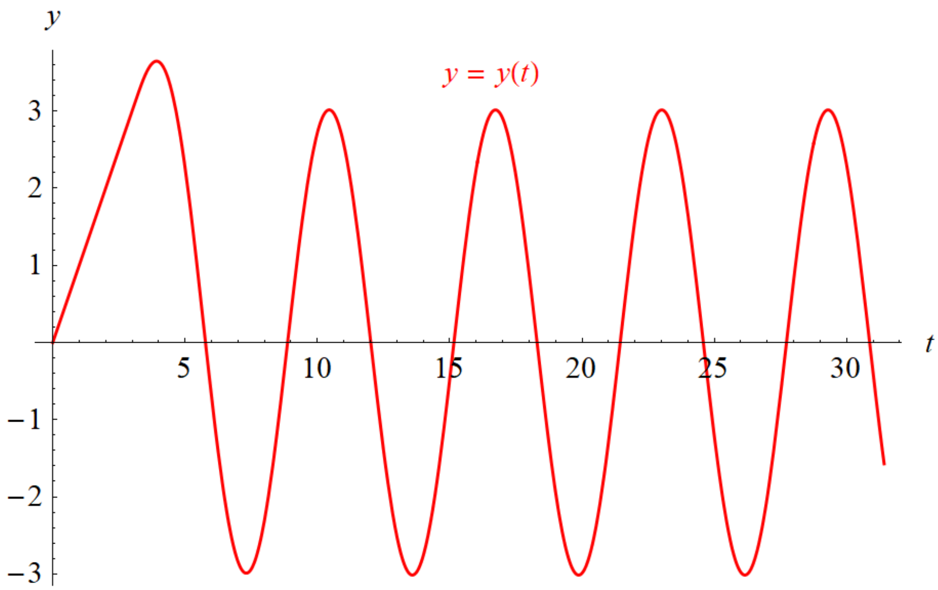
$$\begin{cases} \pi = C_5 \cos\pi + C_6 \sin\pi + \frac{\pi e^\pi}{2} e^{-\pi} \\ 1 = -C_5 \sin\pi + C_6 \cos\pi - \frac{\pi e^\pi}{2} e^{-\pi} \end{cases}$$

$$\begin{cases} \pi = -C_5 + \frac{\pi}{2} \\ 1 = -C_6 - \frac{\pi}{2} \end{cases}$$

$$\begin{cases} C_5 = -\frac{\pi}{2} \\ C_6 = -\frac{\pi}{2} - 1 \end{cases}$$

Therefore,

$$y(t) = \begin{cases} t & \text{if } 0 \le t \le \pi \\ -\frac{\pi}{2}\cos t + \left(-\frac{\pi}{2} - 1\right)\sin t + \frac{\pi e^\pi}{2} e^{-t} & \text{if } t > \pi \end{cases}.$$



The inhomogeneous term is

$$f(t) = \begin{cases} t, & 0 \le t \le \pi \\ \pi e^{\pi - t}, & t > \pi \end{cases}.$$