Problem 6

In each of Problems 5 through 12, find the general solution of the given differential equation. In Problems 11 and 12, g is an arbitrary continuous function.

$$y'' + 9y = 9\sec^2 3t, \qquad 0 < t < \pi/6$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 9y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 9(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 9 = 0$$

$$r = \{-3i, 3i\}$$

Two solutions to equation (1) are then $y_c = e^{-3it}$ and $y_c = e^{3it}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1 e^{-3it} + C_2 e^{3it} \\
&= C_1[\cos(-3t) + i\sin(-3t)] + C_2[\cos(3t) + i\sin(3t)] \\
&= C_1[\cos(3t) - i\sin(3t)] + C_2[\cos(3t) + i\sin(3t)] \\
&= C_1\cos 3t - iC_1 \sin 3t + C_2 \cos 3t + iC_2 \sin 3t \\
&= (C_1 + C_2)\cos 3t + (-iC_1 + iC_2)\sin 3t \\
&= C_3 \cos 3t + C_4 \sin 3t
\end{aligned}$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_3(t)\cos 3t + C_4(t)\sin 3t$$

It satisfies the following ODE.

$$y_p'' + 9y_p = 9\sec^2 3t$$

Substitute the previous formula in for $y_p(t)$.

$$[C_3(t)\cos 3t + C_4(t)\sin 3t]'' + 9[C_3(t)\cos 3t + C_4(t)\sin 3t] = 9\sec^2 3t$$

Evaluate the derivatives.

$$[C_3'(t)\cos 3t - 3C_3(t)\sin 3t + C_4'(t)\sin 3t + 3C_4(t)\cos 3t]' + 9[C_3(t)\cos 3t + C_4(t)\sin 3t] = 9\sec^2 3t$$

$$[C_3''(t)\cos 3t - 3C_3'(t)\sin 3t - 3C_3'(t)\sin 3t - \cancel{9C_3(t)\cos 3t} + C_4''(t)\sin 3t + 3C_4'(t)\cos 3t + 3C_4'(t)\cos 3t - \cancel{9C_4(t)\sin 3t}] \\ + \cancel{9C_3(t)\cos 3t} + \cancel{9C_4(t)\sin 3t} = 9\sec^2 3t$$

$$C_3''(t)\cos 3t - 6C_3'(t)\sin 3t + C_4''(t)\sin 3t + 6C_4'(t)\cos 3t = 9\sec^2 3t$$

If we set

$$C_4''(t)\sin 3t + 6C_4'(t)\cos 3t = 0, \tag{2}$$

then the previous equation reduces to

$$C_3''(t)\cos 3t - 6C_3'(t)\sin 3t = 9\sec^2 3t. \tag{3}$$

The aim now is to solve this system of two equations for $C_3(t)$ and $C_4(t)$. Start by dividing equation (2) by $\sin 3t$.

$$C_4''(t) + 6\frac{\cos 3t}{\sin 3t}C_4'(t) = 0$$

Use an integrating factor I_1 to solve it.

$$I_1 = \exp\left(\int^t 6\frac{\cos 3s}{\sin 3s}\,ds\right) = e^{2\ln\sin 3t} = e^{\ln\sin^2 3t} = \sin^2 3t$$

Multiply both sides of the previous equation by I_1.

$$(\sin^2 3t)C_4''(t) + (6\sin 3t\cos 3t)C_4'(t) = 0$$

The left side can be written as $d/dt[I_1C_4'(t)]$ by the product rule.

$$\frac{d}{dt}[(\sin^2 3t)C_4'(t)] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$(\sin^2 3t)C_4'(t) = 0$$

Divide both sides by $\sin^2 3t$.

$$C_4'(t) = 0$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_4(t) = 0$$

Divide both sides of equation (3) by $\cos 3t$.

$$C_3''(t) - 6\frac{\sin 3t}{\cos 3t}C_3'(t) = 9\sec^3 3t$$

Use an integrating factor I_2 to solve it.

$$I_2 = \exp\left(\int^t -6\frac{\sin 3s}{\cos 3s}\,ds\right) = e^{2\ln\cos 3t} = e^{\ln\cos^2 3t} = \cos^2 3t$$

Multiply both sides of the previous equation by I_2.

$$(\cos^2 3t)C_3''(t) - (6\sin 3t\cos 3t)C_3'(t) = 9\sec 3t$$

The left side can be written as $d/dt[I_2 C_3'(t)]$ by the product rule.

$$\frac{d}{dt}[(\cos^2 3t)C_3'(t)] = 9\sec 3t$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$(\cos^2 3t)C_3'(t) = 9\cdot\frac{1}{3}\ln|\sec 3t + \tan 3t|$$

Since $0 < t < \pi/6$, the absolute value sign can be dropped. Divide both sides by $\cos^2 3t$.

$$C_3'(t) = 3(\sec^2 3t)\ln(\sec 3t + \tan 3t)$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_3(t) = \int^t 3(\sec^2 3s)\ln(\sec 3s + \tan 3s)\, ds$$

Make the following substitution.

$$u = \ln(\sec 3s + \tan 3s) \quad\to\quad e^u = \sec 3s + \tan 3s = \sec 3s + \sqrt{\sec^2 3s - 1} \quad\to\quad \sec 3s = \frac{1+e^{2u}}{2e^u}$$
$$du = 3\sec 3s\, ds$$

As a result,

$$\begin{aligned}
C_3(t) &= \int^{\ln(\sec 3t+\tan 3t)} \frac{1+e^{2u}}{2e^u} u\, du \\
&= \frac{1}{2}\int^{\ln(\sec 3t+\tan 3t)} ue^{-u}\, du + \frac{1}{2}\int^{\ln(\sec 3t+\tan 3t)} ue^{u}\, du \\
&= -\frac{1}{2}e^{-u}(u+1)\Big|^{\ln(\sec 3t+\tan 3t)} + \frac{1}{2}e^{u}(u-1)\Big|^{\ln(\sec 3t+\tan 3t)} \\
&= -\frac{1}{2}e^{-\ln(\sec 3t+\tan 3t)}[\ln(\sec 3t+\tan 3t)+1] + \frac{1}{2}e^{\ln(\sec 3t+\tan 3t)}[\ln(\sec 3t+\tan 3t)-1] \\
&= -\frac{1}{2}e^{\ln(\sec 3t+\tan 3t)^{-1}}[\ln(\sec 3t+\tan 3t)+1] + \frac{1}{2}(\sec 3t+\tan 3t)[\ln(\sec 3t+\tan 3t)-1] \\
&= -\frac{1}{2}(\sec 3t+\tan 3t)^{-1}[\ln(\sec 3t+\tan 3t)+1] + \frac{1}{2}(\sec 3t+\tan 3t)[\ln(\sec 3t+\tan 3t)-1] \\
&= \frac{1}{2}\left[\left(-\frac{1}{\sec 3t+\tan 3t} + \sec 3t + \tan 3t\right)\ln(\sec 3t+\tan 3t) - \frac{1}{\sec 3t+\tan 3t} - (\sec 3t+\tan 3t)\right].
\end{aligned}$$

Consequently, the particular solution is

$$\begin{aligned}
y_p(t) &= C_3(t)\cos 3t + C_4(t)\sin 3t \\
&= \frac{1}{2}\left[\left(-\frac{\cos 3t}{\sec 3t+\tan 3t} + 1 + \sin 3t\right)\ln(\sec 3t + \tan 3t) - \frac{\cos 3t}{\sec 3t + \tan 3t} - (1+\sin 3t)\right] \\
&= \frac{1}{2}\left[\left(-\frac{\cos 3t}{\frac{1}{\cos 3t} + \frac{\sin 3t}{\cos 3t}} + 1 + \sin 3t\right)\ln(\sec 3t + \tan 3t) - \frac{\cos 3t}{\frac{1}{\cos 3t} + \frac{\sin 3t}{\cos 3t}} - (1+\sin 3t)\right] \\
&= \frac{1}{2}\left[\left(-\frac{\cos^2 3t}{1+\sin 3t} + 1 + \sin 3t\right)\ln(\sec 3t + \tan 3t) - \frac{\cos^2 3t}{1+\sin 3t} - (1+\sin 3t)\right] \\
&= \frac{1}{2}\left[\left(\frac{-\cos^2 3t + 1 + 2\sin 3t + \sin^2 3t}{1+\sin 3t}\right)\ln(\sec 3t + \tan 3t) - \frac{\cos^2 3t + 1 + 2\sin 3t + \sin^2 3t}{1+\sin 3t}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(t) &= y_c(t) + y_p(t) \\
&= C_3\cos 3t + C_4 \sin 3t \\
&\qquad + \frac{1}{2}\left[\left(\frac{-\cos^2 3t + 1 + 2\sin 3t + \sin^2 3t}{1+\sin 3t}\right)\ln(\sec 3t + \tan 3t) - \frac{\cos^2 3t + 1 + 2\sin 3t + \sin^2 3t}{1+\sin 3t}\right].
\end{aligned}$$