Problem 9

In each of Problems 5 through 12, find the general solution of the given differential equation. In Problems 11 and 12, g is an arbitrary continuous function.

$$4y'' + y = 2\sec(t/2), \qquad -\pi < t < \pi$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$4y_c'' + y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$4(r^2e^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$4r^2 + 1 = 0$$

$$r = \left\{-\frac{i}{2}, \frac{i}{2}\right\}$$

Two solutions to equation (1) are then $y_c = e^{-it/2}$ and $y_c = e^{it/2}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1e^{-it/2} + C_2e^{it/2} \\
&= C_1\left[\cos\left(-\frac{t}{2}\right) + i\sin\left(-\frac{t}{2}\right)\right] + C_2\left[\cos\left(\frac{t}{2}\right) + i\sin\left(\frac{t}{2}\right)\right] \\
&= C_1\left[\cos\left(\frac{t}{2}\right) - i\sin\left(\frac{t}{2}\right)\right] + C_2\left[\cos\left(\frac{t}{2}\right) + i\sin\left(\frac{t}{2}\right)\right] \\
&= C_1\cos\frac{t}{2} - iC_1\sin\frac{t}{2} + C_2\cos\frac{t}{2} + iC_2\sin\frac{t}{2} \\
&= (C_1 + C_2)\cos\frac{t}{2} + (-iC_1 + iC_2)\sin\frac{t}{2} \\
&= C_3\cos\frac{t}{2} + C_4\sin\frac{t}{2}
\end{aligned}$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_3(t)\cos\frac{t}{2} + C_4(t)\sin\frac{t}{2}$$

It satisfies the following ODE.

$$4y_p'' + y_p = 2\sec(t/2)$$

Substitute the previous formula in for $y_p(t)$.

$$4\left[C_3(t)\cos\frac{t}{2} + C_4(t)\sin\frac{t}{2}\right]'' + \left[C_3(t)\cos\frac{t}{2} + C_4(t)\sin\frac{t}{2}\right] = 2\sec(t/2)$$

Evaluate the derivatives.

$$4\left[C_3'(t)\cos\frac{t}{2} - \frac{1}{2}C_3(t)\sin\frac{t}{2} + C_4'(t)\sin\frac{t}{2} + \frac{1}{2}C_4(t)\cos\frac{t}{2}\right]' + \left[C_3(t)\cos\frac{t}{2} + C_4(t)\sin\frac{t}{2}\right] = 2\sec(t/2)$$

$$\begin{aligned}
4\bigg[C_3''(t)\cos\frac{t}{2} - \frac{1}{2}C_3'(t)\sin\frac{t}{2} - \frac{1}{2}C_3'(t)\sin\frac{t}{2} - \frac{1}{4}C_3(t)\cos\frac{t}{2} \\
+ C_4''(t)\sin\frac{t}{2} + \frac{1}{2}C_4'(t)\cos\frac{t}{2} + \frac{1}{2}C_4'(t)\cos\frac{t}{2} - \frac{1}{4}C_4(t)\sin\frac{t}{2}\bigg] \\
+ \left[C_3(t)\cos\frac{t}{2} + C_4(t)\sin\frac{t}{2}\right] = 2\sec(t/2)
\end{aligned}$$

$$\begin{aligned}
4C_3''(t)\cos\frac{t}{2} - 2C_3'(t)\sin\frac{t}{2} - 2C_3'(t)\sin\frac{t}{2} - \cancel{C_3(t)\cos\frac{t}{2}} \\
+ 4C_4''(t)\sin\frac{t}{2} + 2C_4'(t)\cos\frac{t}{2} + 2C_4'(t)\cos\frac{t}{2} - \cancel{C_4(t)\sin\frac{t}{2}} \\
+ \cancel{C_3(t)\cos\frac{t}{2}} + \cancel{C_4(t)\sin\frac{t}{2}} = 2\sec(t/2)
\end{aligned}$$

$$4C_3''(t)\cos\frac{t}{2} - 4C_3'(t)\sin\frac{t}{2} + 4C_4''(t)\sin\frac{t}{2} + 4C_4'(t)\cos\frac{t}{2} = 2\sec(t/2)$$

Divide both sides by 4.

$$C_3''(t)\cos\frac{t}{2} - C_3'(t)\sin\frac{t}{2} + C_4''(t)\sin\frac{t}{2} + C_4'(t)\cos\frac{t}{2} = \frac{1}{2}\sec\frac{t}{2}$$

If we set

$$C_4''(t)\sin\frac{t}{2} + C_4'(t)\cos\frac{t}{2} = 0, \tag{2}$$

then the previous equation reduces to

$$C_3''(t)\cos\frac{t}{2} - C_3'(t)\sin\frac{t}{2} = \frac{1}{2}\sec\frac{t}{2}. \tag{3}$$

The aim now is to solve this system of two equations for $C_3(t)$ and $C_4(t)$. Start by dividing equation (2) by $\sin(t/2)$.

$$C_4''(t) + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}C_4'(t) = 0$$

Use an integrating factor I_1 to solve it.

$$I_1 = \exp\left(\int^t \frac{\cos\frac{s}{2}}{\sin\frac{s}{2}}\,ds\right) = e^{2\ln\sin(t/2)} = e^{\ln\sin^2(t/2)} = \sin^2\frac{t}{2}$$

Multiply both sides of the previous equation by I_1.

$$\left(\sin^2 \frac{t}{2}\right) C_4''(t) + \sin \frac{t}{2} \cos \frac{t}{2} C_4'(t) = 0$$

The left side can be written as $d/dt[I_1 C_4'(t)]$ by the product rule.

$$\frac{d}{dt}\left[\left(\sin^2 \frac{t}{2}\right) C_4'(t)\right] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$\left(\sin^2 \frac{t}{2}\right) C_4'(t) = 0$$

Divide both sides by $\sin^2(t/2)$.

$$C_4'(t) = 0$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_4(t) = 0$$

Divide both sides of equation (3) by $\cos(t/2)$.

$$C_3''(t) - \frac{\sin \frac{t}{2}}{\cos \frac{t}{2}} C_3'(t) = \frac{1}{2} \sec^2 \frac{t}{2}$$

Use an integrating factor I_2 to solve it.

$$I_2 = \exp\left(\int^t -\frac{\sin \frac{s}{2}}{\cos \frac{s}{2}}\, ds\right) = e^{2 \ln \cos(t/2)} = e^{\ln \cos^2(t/2)} = \cos^2(t/2)$$

Multiply both sides of the previous equation by I_2.

$$\left(\cos^2 \frac{t}{2}\right) C_3''(t) - \left(\sin \frac{t}{2} \cos \frac{t}{2}\right) C_3'(t) = \frac{1}{2}$$

The left side can be written as $d/dt[I_2 C_3'(t)]$ by the product rule.

$$\frac{d}{dt}\left[\left(\cos^2 \frac{t}{2}\right) C_3'(t)\right] = \frac{1}{2}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$\left(\cos^2 \frac{t}{2}\right) C_3'(t) = \frac{t}{2}$$

Divide both sides by $\cos^2(t/2)$.

$$C_3'(t) = \frac{t}{2} \sec^2 \frac{t}{2}$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_3(t) = \int^t \frac{s}{2} \sec^2 \frac{s}{2}\, ds$$

Make the following substitution.

$$u = \frac{s}{2}$$
$$du = \frac{ds}{2} \quad \rightarrow \quad 2\,du = ds$$

As a result,

$$\begin{aligned} C_3(t) &= \int^{t/2} u \sec^2 u \, (2\,du) \\ &= 2 \int^{t/2} u \frac{d}{du}(\tan u)\, du \\ &= 2 \left[u(\tan u) \bigg|^{t/2} - \int^{t/2} (1) \tan u \, du \right] \\ &= 2 \left(\frac{t}{2} \tan \frac{t}{2} - \int^{t/2} \tan u \, du \right) \\ &= 2 \left(\frac{t}{2} \tan \frac{t}{2} + \ln \left| \cos \frac{t}{2} \right| \right) \\ &= t \tan \frac{t}{2} + 2 \ln \left| \cos \frac{t}{2} \right| \\ &= t \tan \frac{t}{2} + \ln \cos^2 \frac{t}{2}. \end{aligned}$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_3(t) \cos \frac{t}{2} + C_4(t) \sin \frac{t}{2} \\ &= t \sin \frac{t}{2} + \left(\cos \frac{t}{2} \right) \ln \cos^2 \frac{t}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_3 \cos \frac{t}{2} + C_4 \sin \frac{t}{2} + t \sin \frac{t}{2} + \left(\cos \frac{t}{2} \right) \ln \cos^2 \frac{t}{2}. \end{aligned}$$