Problem 15

In each of Problems 13 through 20, verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, g is an arbitrary continuous function.

$$ty'' - (1+t)y' + y = t^2 e^{2t}, \quad t > 0; \qquad y_1(t) = 1 + t, \quad y_2(t) = e^t$$

Solution

Verify that the first solution satisfies the associated homogeneous equation.

$$ty_1'' - (1+t)y_1' + y_1 \stackrel{?}{=} 0$$

$$t(1+t)'' - (1+t)(1+t)' + (1+t) \stackrel{?}{=} 0$$

$$-(1+t)(1) + (1+t) \stackrel{?}{=} 0$$

$$0 = 0$$

Now verify that the second solution satisfies the associated homogeneous equation.

$$ty_2'' - (1+t)y_2' + y_2 \stackrel{?}{=} 0$$

$$t(e^t)'' - (1+t)(e^t)' + (e^t) \stackrel{?}{=} 0$$

$$t(e^t) - (1+t)(e^t) + (e^t) \stackrel{?}{=} 0$$

$$te^t - e^t - te^t + e^t \stackrel{?}{=} 0$$

$$0 = 0$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

By the principle of superposition, $y_c(t)$ is a linear combination of $y_1(t)$ and $y_2(t)$.

$$y_c(t) = C_1(1+t) + C_2 e^t$$

According to the method of variation of parameters, the particular solution is found by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)(1+t) + C_2(t)e^t$$

It satisfies the following ODE.

$$ty_p'' - (1+t)y_p' + y_p = t^2 e^{2t}$$

Substitute the previous formula for $y_p(t)$.

$$t[C_1(t)(1+t) + C_2(t)e^t]'' - (1+t)[C_1(t)(1+t) + C_2(t)e^t]' + [C_1(t)(1+t) + C_2(t)e^t] = t^2 e^{2t}$$

Evaluate the derivatives.

$$t[C_1'(t)(1+t)+C_1(t)+C_2'(t)e^t+C_2(t)e^t]' - (1+t)[C_1'(t)(1+t)+C_1(t)+C_2'(t)e^t+C_2(t)e^t] \\ + [C_1(t)(1+t)+C_2(t)e^t] = t^2e^{2t}$$

$$t[C_1''(t)(1+t)+C_1'(t)+C_1'(t)+C_2''(t)e^t+C_2'(t)e^t+C_2'(t)e^t+C_2(t)e^t] \\ - (1+t)[C_1'(t)(1+t)+C_1(t)+C_2'(t)e^t+C_2(t)e^t] + [C_1(t)(1+t)+C_2(t)e^t] = t^2e^{2t}$$

$$t^2C_1''(t)+tC_1''(t)+2tC_1'(t)+tC_2''(t)e^t+2tC_2'(t)e^t+tC_2(t)e^t \\ - C_1'(t)-tC_1'(t)-C_1(t)-C_2'(t)e^t-C_2(t)e^t-tC_1'(t)-t^2C_1'(t)-tC_1(t)-tC_2'(t)e^t-tC_2(t)e^t \\ + C_1(t)+tC_1(t)+C_2(t)e^t = t^2e^{2t}$$

$$t^2C_1''(t)+tC_1''(t)-t^2C_1'(t)-C_1'(t)+te^tC_2''(t)+te^tC_2'(t)-e^tC_2'(t) = t^2e^{2t}$$

If we set

$$t^2C_1''(t)+tC_1''(t)-t^2C_1'(t)-C_1'(t)=0, \tag{1}$$

then the previous equation reduces to

$$te^tC_2''(t)+te^tC_2'(t)-e^tC_2'(t)=t^2e^{2t}. \tag{2}$$

The aim now is to solve this system of equations for $C_1(t)$ and $C_2(t)$. Factor equation (1) and then solve it by integration.

$$t(t+1)C_1''(t)-(t^2+1)C_1'(t)=0$$

$$t(t+1)C_1''(t)=(t^2+1)C_1'(t)$$

$$\frac{C_1''(t)}{C_1'(t)}=\frac{t^2+1}{t(t+1)}$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dt}\ln C_1'(t)=\frac{t^2+1}{t(t+1)}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$\begin{aligned}\ln C_1'(t) &= \int^t \frac{s^2+1}{s(s+1)}\,ds \\ &= \int^t \left(1+\frac{1}{s}-\frac{2}{1+s}\right)ds \\ &= t+\ln t-2\ln(1+t) \\ &= t+\ln t+\ln(1+t)^{-2}\end{aligned}$$

Exponentiate both sides.

$$\begin{aligned}C_1'(t) &= e^{t+\ln t+\ln(1+t)^{-2}} \\ &= e^te^{\ln t}e^{\ln(1+t)^{-2}} \\ &= \frac{te^t}{(1+t)^2}\end{aligned}$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_1(t) = \frac{e^t}{1+t}$$

Divide both sides of equation (2) by te^t.

$$C_2''(t) + \left(1 - \frac{1}{t}\right) C_2'(t) = te^t$$

Use an integrating factor I to solve it.

$$I_1 = \exp\left[\int^t \left(1 - \frac{1}{s}\right) ds\right] = e^{t - \ln t} = e^t e^{\ln t^{-1}} = t^{-1}e^t$$

Multiply both sides of the previous equation by I.

$$t^{-1}e^t C_2''(t) + t^{-1}e^t \left(1 - \frac{1}{t}\right) C_2'(t) = e^{2t}$$

The left side can be written as $d/dt[IC_2'(t)]$ by the product rule.

$$\frac{d}{dt}[t^{-1}e^t C_2'(t)] = e^{2t}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$t^{-1}e^t C_2'(t) = \frac{1}{2}e^{2t}$$

Divide both sides by $t^{-1}e^t$.

$$C_2'(t) = \frac{t}{2}e^t$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_2(t) = \frac{1}{2}e^t(t-1)$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_1(t)y_1(t) + C_2(t)y_2(t) \\ &= C_1(t)(1+t) + C_2(t)e^t \\ &= \frac{e^t}{1+t}(1+t) + \frac{1}{2}e^t(t-1)e^t \\ &= e^t + \frac{1}{2}e^{2t}(t-1). \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1(1+t) + C_2e^t + e^t + \frac{1}{2}e^{2t}(t-1) \\ &= C_1(1+t) + (C_2+1)e^t + \frac{1}{2}e^{2t}(t-1) \\ &= C_1(1+t) + C_3e^t + \frac{1}{2}e^{2t}(t-1), \end{aligned}$$

where a new constant C_3 was used for $C_2 + 1$.