Problem 16

In each of Problems 13 through 20, verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, g is an arbitrary continuous function.

$$(1-t)y'' + ty' - y = 2(t-1)^2e^{-t}, \quad 0 < t < 1; \qquad y_1(t) = e^t, \quad y_2(t) = t$$

Solution

Verify that the first solution satisfies the associated homogeneous equation.

$$(1-t)y_1'' + ty_1' - y_1 \stackrel{?}{=} 0$$

$$(1-t)(e^t)'' + t(e^t)' - (e^t) \stackrel{?}{=} 0$$

$$(1-t)(e^t) + t(e^t) - (e^t) \stackrel{?}{=} 0$$

$$e^t - te^t + te^t - e^t \stackrel{?}{=} 0$$

$$0 = 0$$

Now verify that the second solution satisfies the associated homogeneous equation.

$$(1-t)y_2'' + ty_2' - y_2 \stackrel{?}{=} 0$$

$$(1-t)(t)'' + t(t)' - (t) \stackrel{?}{=} 0$$

$$t - t \stackrel{?}{=} 0$$

$$0 = 0$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

By the principle of superposition, $y_c(t)$ is a linear combination of $y_1(t)$ and $y_2(t)$.

$$y_c(t) = C_1e^t + C_2t$$

According to the method of variation of parameters, the particular solution is found by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)e^t + C_2(t)t$$

It satisfies the following ODE.

$$(1-t)y_p'' + ty_p' - y_p = 2(t-1)^2e^{-t}$$

Substitute the previous formula for $y_p(t)$.

$$(1-t)[C_1(t)e^t + C_2(t)t]'' + t[C_1(t)e^t + C_2(t)t]' - [C_1(t)e^t + C_2(t)t] = 2(t-1)^2e^{-t}$$

Evaluate the derivatives.

$$(1-t)[C_1'(t)e^t + C_1(t)e^t + C_2'(t)t + C_2(t)]' + t[C_1'(t)e^t + C_1(t)e^t + C_2'(t)t + C_2(t)] \\ - [C_1(t)e^t + C_2(t)t] = 2(t-1)^2e^{-t}$$

$$(1-t)[C_1''(t)e^t + C_1'(t)e^t + C_1'(t)e^t + C_1(t)e^t + C_2''(t)t + C_2'(t) + C_2'(t)] \\ + t[C_1'(t)e^t + C_1(t)e^t + C_2'(t)t + C_2(t)] - [C_1(t)e^t + C_2(t)t] = 2(t-1)^2e^{-t}$$

$$e^tC_1''(t) - te^tC_1''(t) - te^tC_1'(t) + 2e^tC_1'(t) + tC_2''(t) - t^2C_2''(t) + t^2C_2'(t) - 2tC_2'(t) + 2C_2'(t) = 2(t-1)^2e^{-t}$$

If we set

$$tC_2''(t) - t^2C_2''(t) + t^2C_2'(t) - 2tC_2'(t) + 2C_2'(t) = 0, \tag{1}$$

then the previous equation reduces to

$$e^tC_1''(t) - te^tC_1''(t) - te^tC_1'(t) + 2e^tC_1'(t) = 2(t-1)^2e^{-t}. \tag{2}$$

The aim now is to solve this system of equations for $C_1(t)$ and $C_2(t)$. Factor equation (1) and then divide both sides of it by $t - t^2$.

$$C_2''(t) + \frac{t^2 - 2t + 2}{t - t^2}C_2'(t) = 0$$

Use an integrating factor I_1 to solve it.

$$I_1 = \exp\left(\int^t \frac{s^2 - 2s + 2}{s - s^2}\,ds\right) = \exp\left[\int^t \left(\frac{2}{s} + \frac{s}{1-s}\right)ds\right] = e^{2\ln t - t - \ln(1-t)} = e^{\ln t^2}e^{-t}e^{\ln(1-t)^{-1}} = \frac{t^2}{1-t}e^{-t}$$

Multiply both sides of the previous equation by I_1.

$$\frac{t^2}{1-t}e^{-t}C_2''(t) + \frac{te^{-t}(t^2 - 2t + 2)}{(1-t)^2}C_2'(t) = 0$$

The left side can be written as $d/dt[I_1C_2'(t)]$ by the product rule.

$$\frac{d}{dt}\left[\frac{t^2}{1-t}e^{-t}C_2'(t)\right] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$\frac{t^2}{1-t}e^{-t}C_2'(t) = 0$$

Divide both sides by $t^2(1-t)^{-1}e^{-t}$.

$$C_2'(t) = 0$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_2(t) = 0$$

Factor the left side of equation (2)

$$e^t(1-t)C_1''(t) + e^t(2-t)C_1'(t) = 2(1-t)^2e^{-t}$$

and then divide both sides by $e^t(1-t)$.

$$C_1''(t) + \frac{2-t}{1-t}C_1'(t) = 2(1-t)e^{-2t}$$

Use another integrating factor I_2 to solve it.

$$I_2 = \exp\left(\int^t \frac{2-s}{1-s}\,ds\right) = \exp\left[\int^t \left(\frac{2}{1-s} - \frac{s}{1-s}\right) ds\right] = e^{-2\ln(1-t)+t+\ln(1-t)} = e^{\ln(1-t)^{-1}}e^t = \frac{e^t}{1-t}$$

Multiply both sides of the previous equation by I_2.

$$\frac{e^t}{1-t}C_1''(t) + \frac{2-t}{(1-t)^2}e^tC_1'(t) = 2e^{-t}$$

The left side can be written as $d/dt[I_2C_1'(t)]$ by the product rule.

$$\frac{d}{dt}\left[\frac{e^t}{1-t}C_1'(t)\right] = 2e^{-t}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$\frac{e^t}{1-t}C_1'(t) = -2e^{-t}$$

Divide both sides by $e^t(1-t)^{-1}$.

$$\begin{aligned} C_1'(t) &= -2(1-t)e^{-2t} \\ &= -2e^{-2t} + 2te^{-2t} \end{aligned}$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$\begin{aligned} C_1(t) &= e^{-2t} - \frac{1}{2}e^{-2t} - te^{-2t} \\ &= \frac{1}{2}e^{-2t} - te^{-2t} \end{aligned}$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_1(t)y_1(t) + C_2(t)y_2(t) \\ &= C_1(t)e^t + C_2(t)t \\ &= \left(\frac{1}{2}e^{-2t} - te^{-2t}\right)e^t \\ &= \frac{1}{2}e^{-t} - te^{-t} \\ &= -\frac{1}{2}(2t-1)e^{-t}. \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1e^t + C_2t - \frac{1}{2}(2t-1)e^{-t}. \end{aligned}$$