Problem 17

In each of Problems 13 through 20, verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, g is an arbitrary continuous function.

$$x^2y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0; \qquad y_1(x) = x^2, \quad y_2(x) = x^2 \ln x$$

Solution

Verify that the first solution satisfies the associated homogeneous equation.

$$x^2y_1'' - 3xy_1' + 4y_1 \stackrel{?}{=} 0$$

$$x^2(x^2)'' - 3x(x^2)' + 4(x^2) \stackrel{?}{=} 0$$

$$x^2(2) - 3x(2x) + 4(x^2) \stackrel{?}{=} 0$$

$$2x^2 - 6x^2 + 4x^2 \stackrel{?}{=} 0$$

$$0 = 0$$

Now verify that the second solution satisfies the associated homogeneous equation.

$$x^2y_2'' - 3xy_2' + 4y_2 \stackrel{?}{=} 0$$

$$x^2(x^2 \ln x)'' - 3x(x^2 \ln x)' + 4(x^2 \ln x) \stackrel{?}{=} 0$$

$$x^2(2 \ln x + 2 + 1) - 3x(2x \ln x + x) + 4(x^2 \ln x) \stackrel{?}{=} 0$$

$$2x^2 \ln x + 3x^2 - 6x^2 \ln x - 3x^2 + 4x^2 \ln x \stackrel{?}{=} 0$$

$$0 = 0$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(x)$ and the particular solution $y_p(x)$.

$$y(x) = y_c(x) + y_p(x)$$

By the principle of superposition, $y_c(x)$ is a linear combination of $y_1(x)$ and $y_2(x)$.

$$y_c(x) = C_1x^2 + C_2x^2 \ln x$$

According to the method of variation of parameters, the particular solution is found by allowing the parameters in $y_c(x)$ to vary.

$$y_p(x) = C_1(x)x^2 + C_2(x)x^2 \ln x$$

It satisfies the following ODE.

$$x^2y_p'' - 3xy_p' + 4y_p = x^2 \ln x$$

Substitute the previous formula for $y_p(x)$.

$$x^2[C_1(x)x^2 + C_2(x)x^2\ln x]'' - 3x[C_1(x)x^2 + C_2(x)x^2\ln x]' + 4[C_1(x)x^2 + C_2(x)x^2\ln x] = x^2\ln x$$

Evaluate the derivatives.

$$\begin{aligned} x^2[C_1'(x)x^2 + 2C_1(x)x + C_2'(x)x^2\ln x + 2C_2(x)x\ln x + C_2(x)x]' \\ - 3x[C_1'(x)x^2 + 2C_1(x)x + C_2'(x)x^2\ln x + 2C_2(x)x\ln x + C_2(x)x] \\ + 4[C_1(x)x^2 + C_2(x)x^2\ln x] = x^2\ln x \end{aligned}$$

$$\begin{aligned} x^2[C_1''(x)x^2 + 2C_1'(x)x + 2C_1'(x)x + 2C_1(x) + C_2''(x)x^2\ln x + 2C_2'(x)x\ln x + C_2'(x)x \\ + 2C_2'(x)x\ln x + 2C_2(x)\ln x + 2C_2(x) + C_2'(x)x + C_2(x)] \\ - 3x[C_1'(x)x^2 + 2C_1(x)x + C_2'(x)x^2\ln x + 2C_2(x)x\ln x + C_2(x)x] \\ + 4[C_1(x)x^2 + C_2(x)x^2\ln x] = x^2\ln x \end{aligned}$$

$$x^4C_1''(x) + x^3C_1'(x) + (x^4\ln x)C_2''(x) + (x^3\ln x)C_2'(x) + 2x^3C_2'(x) = x^2\ln x$$

If we set

$$x^4C_1''(x) + x^3C_1'(x) = 0, \tag{1}$$

then the previous equation reduces to

$$(x^4\ln x)C_2''(x) + (x^3\ln x)C_2'(x) + 2x^3C_2'(x) = x^2\ln x. \tag{2}$$

The aim now is to solve this system of equations for $C_1(x)$ and $C_2(x)$. Divide both sides of equation (1) by x^4.

$$C_1''(x) + \frac{1}{x}C_1'(x) = 0$$

Use an integrating factor I_1 to solve it.

$$I_1 = \exp\left(\int^x \frac{1}{s}\,ds\right) = e^{\ln x} = x$$

Multiply both sides of the previous equation by I_1.

$$xC_1''(x) + C_1'(x) = 0$$

The left side can be written as $d/dx[I_1C_1'(x)]$ by the product rule.

$$\frac{d}{dx}[xC_1'(x)] = 0$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$xC_1'(x) = 0$$

Divide both sides by x.

$$C_1'(x) = 0$$

Integrate both sides with respect to x once more, setting the integration constant to zero.

$$C_1(x) = 0$$

Factor the left side of equation (2)

$$(x^4 \ln x)C_2''(x) + (x^3 \ln x + 2x^3)C_2'(x) = x^2 \ln x$$

and then divide both sides by $x^4 \ln x$.

$$C_2''(x) + \left(\frac{1}{x} + \frac{2}{x \ln x}\right) C_2'(x) = \frac{1}{x^2}$$

Use another integrating factor I_2 to solve it.

$$I_2 = \exp\left[\int^x \left(\frac{1}{s} + \frac{2}{s \ln s}\right) ds\right] = e^{\ln x + 2 \ln \ln x} = e^{\ln x} e^{\ln(\ln x)^2} = x(\ln x)^2$$

Multiply both sides of the previous equation by I_2.

$$x(\ln x)^2 C_2''(x) + [(\ln x)^2 + 2 \ln x]C_2'(x) = \frac{(\ln x)^2}{x}$$

The left side can be written as $d/dx[I_2 C_2'(x)]$ by the product rule.

$$\frac{d}{dx}[x(\ln x)^2 C_2'(x)] = \frac{(\ln x)^2}{x}$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$x(\ln x)^2 C_2'(x) = \frac{(\ln x)^3}{3}$$

Divide both sides by $x(\ln x)^2$.

$$C_2'(x) = \frac{\ln x}{3x}$$

Integrate both sides with respect to x once more, setting the integration constant to zero.

$$C_2(x) = \frac{(\ln x)^2}{6}$$

The particular solution is then

$$\begin{aligned} y_p(x) &= C_1(x)y_1(x) + C_2(x)y_2(x) \\ &= C_1(x)x^2 + C_2(x)x^2 \ln x \\ &= \left[\frac{(\ln x)^2}{6}\right] x^2 \ln x \\ &= \frac{x^2(\ln x)^3}{6}. \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= C_1 x^2 + C_2 x^2 \ln x + \frac{x^2(\ln x)^3}{6}. \end{aligned}$$