

### Problem 18

In each of Problems 13 through 20, verify that the given functions  $y_1$  and  $y_2$  satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20,  $g$  is an arbitrary continuous function.

$$x^2 y'' + xy' + (x^2 - 0.25)y = 3x^{3/2} \sin x, \quad x > 0; \quad y_1(x) = x^{-1/2} \sin x, \quad y_2(x) = x^{-1/2} \cos x$$

#### Solution

Verify that the first solution satisfies the associated homogeneous equation.

$$x^2 y_1'' + xy_1' + (x^2 - 0.25)y_1 \stackrel{?}{=} 0$$

$$x^2(x^{-1/2} \sin x)'' + x(x^{-1/2} \sin x)' + (x^2 - 0.25)(x^{-1/2} \sin x) \stackrel{?}{=} 0$$

$$x^2 \left( -\frac{1}{2}x^{-3/2} \sin x + x^{-1/2} \cos x \right)' + x \left( -\frac{1}{2}x^{-3/2} \sin x + x^{-1/2} \cos x \right) + (x^2 - 0.25)(x^{-1/2} \sin x) \stackrel{?}{=} 0$$

$$x^2 \left( \frac{3}{4}x^{-5/2} \sin x - \frac{1}{2}x^{-3/2} \cos x - \frac{1}{2}x^{-3/2} \cos x - x^{-1/2} \sin x \right) + x \left( -\frac{1}{2}x^{-3/2} \sin x + x^{-1/2} \cos x \right) + (x^2 - 0.25)(x^{-1/2} \sin x) \stackrel{?}{=} 0$$

$$\cancel{\frac{3}{4}x^{-5/2} \sin x} - \cancel{\frac{1}{2}x^{-3/2} \cos x} - \cancel{\frac{1}{2}x^{-3/2} \cos x} - \cancel{x^{-1/2} \sin x} - \cancel{\frac{1}{2}x^{-3/2} \sin x} + \cancel{x^{-1/2} \cos x} + \cancel{x^{3/2} \sin x} - \cancel{0.25x^{-1/2} \sin x} \stackrel{?}{=} 0$$

$$0 = 0$$

Now verify that the second solution satisfies the associated homogeneous equation.

$$x^2 y_2'' + xy_2' + (x^2 - 0.25)y_2 \stackrel{?}{=} 0$$

$$x^2(x^{-1/2} \cos x)'' + x(x^{-1/2} \cos x)' + (x^2 - 0.25)(x^{-1/2} \cos x) \stackrel{?}{=} 0$$

$$x^2 \left( -\frac{1}{2}x^{-3/2} \cos x - x^{-1/2} \sin x \right)' + x \left( -\frac{1}{2}x^{-3/2} \cos x - x^{-1/2} \sin x \right) + (x^2 - 0.25)(x^{-1/2} \cos x) \stackrel{?}{=} 0$$

$$x^2 \left( \frac{3}{4}x^{-5/2} \cos x + \frac{1}{2}x^{-3/2} \sin x + \frac{1}{2}x^{-3/2} \sin x - x^{-1/2} \cos x \right) + x \left( -\frac{1}{2}x^{-3/2} \cos x - x^{-1/2} \sin x \right) + (x^2 - 0.25)(x^{-1/2} \cos x) \stackrel{?}{=} 0$$

$$\cancel{\frac{3}{4}x^{-5/2} \cos x} + \cancel{\frac{1}{2}x^{-3/2} \sin x} + \cancel{\frac{1}{2}x^{-3/2} \sin x} - \cancel{x^{-1/2} \cos x} - \cancel{\frac{1}{2}x^{-3/2} \cos x} - \cancel{x^{-1/2} \sin x} + \cancel{x^{3/2} \cos x} - \cancel{0.25x^{-1/2} \cos x} \stackrel{?}{=} 0$$

$$0 = 0$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution  $y_c(x)$  and the particular solution  $y_p(x)$ .

$$y(x) = y_c(x) + y_p(x)$$

By the principle of superposition,  $y_c(x)$  is a linear combination of  $y_1(x)$  and  $y_2(x)$ .

$$y_c(x) = C_1 x^{-1/2} \sin x + C_2 x^{-1/2} \cos x$$

According to the method of variation of parameters, the particular solution is found by allowing the parameters in  $y_c(x)$  to vary.

$$y_p(x) = C_1(x)x^{-1/2} \sin x + C_2(x)x^{-1/2} \cos x$$

It satisfies the following ODE.

$$x^2 y_p'' + x y_p' + (x^2 - 0.25)y_p = 3x^{3/2} \sin x$$

Substitute the previous formula for  $y_p(x)$ .

$$\begin{aligned} x^2 [C_1(x)x^{-1/2} \sin x + C_2(x)x^{-1/2} \cos x]'' + x [C_1(x)x^{-1/2} \sin x + C_2(x)x^{-1/2} \cos x]' \\ + (x^2 - 0.25)[C_1(x)x^{-1/2} \sin x + C_2(x)x^{-1/2} \cos x] = 3x^{3/2} \sin x \end{aligned}$$

Evaluate the derivatives.

$$\begin{aligned} x^2 \left[ C_1'(x)x^{-1/2} \sin x - \frac{1}{2}C_1(x)x^{-3/2} \sin x + C_1(x)x^{-1/2} \cos x + C_2'(x)x^{-1/2} \cos x \right. \\ \left. - \frac{1}{2}C_2(x)x^{-3/2} \cos x - C_2(x)x^{-1/2} \sin x \right]' \\ + x \left[ C_1'(x)x^{-1/2} \sin x - \frac{1}{2}C_1(x)x^{-3/2} \sin x + C_1(x)x^{-1/2} \cos x + C_2'(x)x^{-1/2} \cos x \right. \\ \left. - \frac{1}{2}C_2(x)x^{-3/2} \cos x - C_2(x)x^{-1/2} \sin x \right] \\ + (x^2 - 0.25)[C_1(x)x^{-1/2} \sin x + C_2(x)x^{-1/2} \cos x] = 3x^{3/2} \sin x \end{aligned}$$

$$\begin{aligned} x^2 \left[ C_1''(x)x^{-1/2} \sin x - \frac{1}{2}C_1'(x)x^{-3/2} \sin x + C_1'(x)x^{-1/2} \cos x - \frac{1}{2}C_1'(x)x^{-3/2} \sin x + \frac{3}{4}C_1(x)x^{-5/2} \sin x \right. \\ \left. - \frac{1}{2}C_1(x)x^{-3/2} \cos x + C_1'(x)x^{-1/2} \cos x - \frac{1}{2}C_1(x)x^{-3/2} \cos x - C_1(x)x^{-1/2} \sin x \right. \\ \left. + C_2''(x)x^{-1/2} \cos x - \frac{1}{2}C_2'(x)x^{-3/2} \cos x - C_2'(x)x^{-1/2} \sin x - \frac{1}{2}C_2'(x)x^{-3/2} \cos x \right. \\ \left. + \frac{3}{4}C_2(x)x^{-5/2} \cos x + \frac{1}{2}C_2(x)x^{-3/2} \sin x - C_2'(x)x^{-1/2} \sin x + \frac{1}{2}C_2(x)x^{-3/2} \sin x - C_2(x)x^{-1/2} \cos x \right] \\ + x \left[ C_1'(x)x^{-1/2} \sin x - \frac{1}{2}C_1(x)x^{-3/2} \sin x + C_1(x)x^{-1/2} \cos x + C_2'(x)x^{-1/2} \cos x \right. \\ \left. - \frac{1}{2}C_2(x)x^{-3/2} \cos x - C_2(x)x^{-1/2} \sin x \right] \\ + (x^2 - 0.25)[C_1(x)x^{-1/2} \sin x + C_2(x)x^{-1/2} \cos x] = 3x^{3/2} \sin x \end{aligned}$$

Simplify the left side.

$$x^{3/2}(\sin x)C_1''(x) + 2x^{3/2}(\cos x)C_1'(x) + x^{3/2}(\cos x)C_2''(x) - 2x^{3/2}(\sin x)C_2'(x) = 3x^{3/2} \sin x$$

Divide both sides by  $x^{3/2}$ .

$$(\sin x)C_1''(x) + 2(\cos x)C_1'(x) + (\cos x)C_2''(x) - 2(\sin x)C_2'(x) = 3 \sin x$$

If we set

$$(\cos x)C_2''(x) - 2(\sin x)C_2'(x) = 0, \quad (1)$$

then the previous equation reduces to

$$(\sin x)C_1''(x) + 2(\cos x)C_1'(x) = 3 \sin x. \quad (2)$$

The aim now is to solve this system of equations for  $C_1(x)$  and  $C_2(x)$ . Divide both sides of equation (1) by  $\cos x$ .

$$C_2''(x) - 2\frac{\sin x}{\cos x}C_2'(x) = 0$$

Use an integrating factor  $I_1$  to solve it.

$$I_1 = \exp\left(\int^x -2\frac{\sin s}{\cos s} ds\right) = e^{2\ln \cos x} = e^{\ln \cos^2 x} = \cos^2 x$$

Multiply both sides of the previous equation by  $I_1$ .

$$(\cos^2 x)C_2''(x) - (2 \sin x \cos x)C_2'(x) = 0$$

The left side can be written as  $d/dx[I_1 C_2'(x)]$  by the product rule.

$$\frac{d}{dx}[(\cos^2 x)C_2'(x)] = 0$$

Integrate both sides with respect to  $x$ , setting the integration constant to zero.

$$(\cos^2 x)C_2'(x) = 0$$

Divide both sides by  $\cos^2 x$ .

$$C_2'(x) = 0$$

Integrate both sides with respect to  $x$  once more, setting the integration constant to zero.

$$C_2(x) = 0$$

Divide both sides of equation (2) by  $\sin x$ .

$$C_1''(x) + 2\frac{\cos x}{\sin x}C_1'(x) = 3$$

Use another integrating factor  $I_2$  to solve it.

$$I_2 = \exp\left(\int^x 2\frac{\cos s}{\sin s} ds\right) = e^{2\ln \sin x} = e^{\ln \sin^2 x} = \sin^2 x$$

Multiply both sides of the previous equation by  $I_2$ .

$$(\sin^2 x)C_1''(x) + (2 \cos x \sin x)C_1'(x) = 3 \sin^2 x$$

The left side can be written as  $d/dx[I_2C_1'(x)]$  by the product rule.

$$\frac{d}{dx}[(\sin^2 x)C_1'(x)] = \frac{3}{2}(1 - \cos 2x)$$

Integrate both sides with respect to  $x$ , setting the integration constant to zero.

$$\begin{aligned}(\sin^2 x)C_1'(x) &= \frac{3}{2}x - \frac{3}{4}\sin 2x \\ &= \frac{3}{2}x - \frac{3}{2}\sin x \cos x\end{aligned}$$

Divide both sides by  $\sin^2 x$ .

$$C_1'(x) = \frac{3}{2}x \csc^2 x - \frac{3}{2}\cot x$$

Integrate both sides with respect to  $x$  once more, setting the integration constant to zero.

$$\begin{aligned}C_1(x) &= \int^x \left( \frac{3}{2}s \csc^2 s - \frac{3}{2}\cot s \right) ds \\ &= \frac{3}{2} \int^x s \csc^2 s ds - \frac{3}{2} \int^x \cot s ds \\ &= \frac{3}{2} \int^x s \frac{d}{ds}(-\cot s) ds - \frac{3}{2} \int^x \cot s ds \\ &= \frac{3}{2} \left[ s(-\cot s) \Big|_0^x - \int_0^x (1)(-\cot s) ds \right] - \frac{3}{2} \int^x \cot s ds \\ &= \frac{3}{2} \left( -x \cot x + \int^x \cot s ds \right) - \frac{3}{2} \int^x \cot s ds \\ &= -\frac{3}{2}x \cot x\end{aligned}$$

The particular solution is then

$$\begin{aligned}y_p(x) &= C_1(x)y_1(x) + C_2(x)y_2(x) \\ &= C_1(x)x^{-1/2}\sin x + C_2(x)x^{-1/2}\cos x \\ &= \left( -\frac{3}{2}x \cot x \right) x^{-1/2}\sin x \\ &= -\frac{3}{2}x^{1/2}\cos x.\end{aligned}$$

Therefore, the general solution is

$$\begin{aligned}y(x) &= y_c(x) + y_p(x) \\ &= C_1x^{-1/2}\sin x + C_2x^{-1/2}\cos x - \frac{3}{2}x^{1/2}\cos x.\end{aligned}$$