Problem 23

(a) Use the result of Problem 22 to show that the solution of the initial value problem

$$y'' + y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0 \tag{i}$$

is

$$y = \int_{t_0}^{t} \sin(t - s) g(s)\, ds. \tag{ii}$$

(b) Use the result of Problem 21 to find the solution of the initial value problem

$$y'' + y = g(t), \qquad y(0) = y_0, \quad y'(0) = y_0'.$$

Solution

Part (a)

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 1 = 0$$

$$r = \{-i, i\}$$

Two solutions to equation (1) are then $y_c = e^{-it}$ and $y_c = e^{it}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1 e^{-it} + C_2 e^{it} \\
&= C_1[\cos(-t) + i\sin(-t)] + C_2[\cos(t) + i\sin(t)] \\
&= C_1[\cos(t) - i\sin(t)] + C_2[\cos(t) + i\sin(t)] \\
&= C_1 \cos t - iC_1 \sin t + C_2 \cos t + iC_2 \sin t \\
&= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t \\
&= C_3 \cos t + C_4 \sin t
\end{aligned}$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_3(t)\cos t + C_4(t)\sin t$$

It satisfies the following ODE.

$$y_p'' + y_p = g(t)$$

Plug in the previous formula for $y_p(t)$.

$$[C_3(t)\cos t + C_4(t)\sin t]'' + [C_3(t)\cos t + C_4(t)\sin t] = g(t)$$

Evaluate the derivatives.

$$[C_3'(t)\cos t - C_3(t)\sin t + C_4'(t)\sin t + C_4(t)\cos t]' + [C_3(t)\cos t + C_4(t)\sin t] = g(t)$$

$$[C_3''(t)\cos t - C_3'(t)\sin t - C_3'(t)\sin t - \cancel{C_3(t)\cos t} + C_4''(t)\sin t + C_4'(t)\cos t + C_4'(t)\cos t - \cancel{C_4(t)\sin t}] + [\cancel{C_3(t)\cos t} + \cancel{C_4(t)\sin t}] = g(t)$$

$$C_3''(t)\cos t - 2C_3'(t)\sin t + C_4''(t)\sin t + 2C_4'(t)\cos t = g(t)$$

If we set

$$C_3''(t)\cos t - C_3'(t)\sin t + C_4''(t)\sin t + C_4'(t)\cos t = 0, \tag{2}$$

then the previous equation reduces to

$$-C_3'(t)\sin t + C_4'(t)\cos t = g(t). \tag{3}$$

The aim now is to solve this system of two equations for $C_3(t)$ and $C_4(t)$. Start by rewriting equation (2).

$$\frac{d}{dt}[C_3'(t)\cos t] + \frac{d}{dt}[C_4'(t)\sin t] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_3'(t)\cos t + C_4'(t)\sin t = 0$$

Solve for $C_3'(t)$.

$$C_3'(t) = -\frac{\sin t}{\cos t}C_4'(t) \tag{4}$$

Substitute this formula into equation (3).

$$-\left[-\frac{\sin t}{\cos t}C_4'(t)\right]\sin t + C_4'(t)\cos t = g(t)$$

$$C_4'(t)\left(\frac{\sin^2 t}{\cos t} + \cos t\right) = g(t)$$

$$C_4'(t)\left(\frac{\sin^2 t + \cos^2 t}{\cos t}\right) = g(t)$$

$$C_4'(t)\left(\frac{1}{\cos t}\right) = g(t)$$

Multiply both sides by $\cos t$.

$$C_4'(t) = g(t)\cos t$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_4(t) = \int^t g(s)\cos s\,ds$$

Substitute the previous formula for $C_4'(t)$ into equation (4).

$$\begin{aligned} C_3'(t) &= -\frac{\sin t}{\cos t}[g(t)\cos t] \\ &= -g(t)\sin t \end{aligned}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_3(t) = -\int^t g(s)\sin s\,ds$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_3(t)\cos t + C_4(t)\sin t \\ &= \cos t\left[-\int^t g(s)\sin s\,ds\right] + \sin t\left[\int^t g(s)\cos s\,ds\right] \\ &= -\cos t\int^t g(s)\sin s\,ds + \sin t\int^t g(s)\cos s\,ds \\ &= \int^t [-g(s)\sin s\cos t + g(s)\sin t\cos s]\,ds \\ &= \int^t (\sin t\cos s - \sin s\cos t)g(s)\,ds \\ &= \int^t \sin(t-s)g(s)\,ds \\ &= \int_{t_0}^t \sin(t-s)g(s)\,ds. \end{aligned}$$

The lower limit of integration is arbitrary and has been set to t_0, the value of t that the initial conditions are given for. Consequently, the general solution is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_3\cos t + C_4\sin t + \int_{t_0}^t \sin(t-s)g(s)\,ds. \end{aligned}$$

Use the Leibnitz rule,

$$\frac{d}{dt}\int_{j(t)}^{h(t)} f(t,s)\,ds = \int_{j(t)}^{h(t)} \frac{\partial}{\partial t}f(t,s)\,ds + \frac{dh}{dt}f[t,h(t)] - \frac{dj}{dt}f[t,j(t)],$$

to differentiate the general solution.

$$\begin{aligned} y'(t) &= -C_3\sin t + C_4\cos t + \int_{t_0}^t \frac{\partial}{\partial t}\sin(t-s)g(s)\,ds + 1\cdot\sin(0)g(t) \\ &= -C_3\sin t + C_4\cos t + \int_{t_0}^t \cos(t-s)g(s)\,ds \end{aligned}$$

Apply the initial conditions now to determine C_3 and C_4.

$$\begin{aligned} y(t_0) &= C_3 \cos t_0 + C_4 \sin t_0 = 0 \\ y'(t_0) &= -C_3 \sin t_0 + C_4 \cos t_0 = 0 \end{aligned}$$

This system is satisfied only if $C_3 = 0$ and $C_4 = 0$. Therefore,

$$y(t) = \int_{t_0}^{t} \sin(t-s) g(s)\, ds.$$

Part (b)

$$y'' + y = g(t), \qquad y(0) = y_0, \quad y'(0) = y_0'.$$

Make the substitution, $y(t) = u(t) + v(t)$.

$$[u(t) + v(t)]'' + [u(t) + v(t)] = g(t)$$

Evaluate the derivatives.

$$u''(t) + v''(t) + u(t) + v(t) = g(t)$$

If we set

$$u''(t) + u(t) = 0, \tag{5}$$

then the previous equation reduces to

$$v''(t) + v(t) = g(t). \tag{6}$$

Apply the substitution now to the initial conditions.

$$\begin{aligned} y(0) = y_0 \qquad &\rightarrow \qquad u(0) + v(0) = y_0 \\ y'(0) = y_0' \qquad &\rightarrow \qquad u'(0) + v'(0) = y_0' \end{aligned}$$

We will set $u(0) = y_0$ and $u'(0) = y_0'$ so that the initial conditions for v are homogeneous.

$$\begin{aligned} u(0) + v(0) = y_0 \qquad &\rightarrow \qquad y_0 + v(0) = y_0 \qquad \rightarrow \qquad v(0) = 0 \\ u'(0) + v'(0) = y_0' \qquad &\rightarrow \qquad y_0' + v'(0) = y_0' \qquad \rightarrow \qquad v'(0) = 0 \end{aligned}$$

Equation (5) is identical to equation (1), so the general solution is the same.

$$u(t) = C_5 \cos t + C_6 \sin t$$

Take a derivative of it with respect to t.

$$u'(t) = -C_5 \sin t + C_6 \cos t$$

Apply the initial conditions now to determine C_5 and C_6.

$$\begin{aligned} u(0) &= C_5 = y_0 \\ u'(0) &= C_6 = y_0' \end{aligned}$$

Consequently,

$$u(t) = y_0 \cos t + y_0' \sin t.$$

The solution to $v(t)$ is the same as the result from part (a).

$$v(t) = \int_{t_0}^{t} \sin(t - s) g(s)\, ds$$

Therefore,

$$\begin{aligned} y(t) &= u(t) + v(t) \\ &= y_0 \cos t + y_0' \sin t + \int_{t_0}^{t} \sin(t - s) g(s)\, ds. \end{aligned}$$