Problem 26

Use the result of Problem 22 to find the solution of the initial value problem

$$L[y] = (D-a)^2 y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where a is any real number.

Solution

Method Using Operator Factorization

Since the operator has been factored, the substitution $u = (D-a)y$ is invited. The ODE then becomes

$$(D-a)u = g(t).$$

As a result, the second-order ODE has been reduced to a system of (decoupled) first-order ODEs.

$$\begin{cases} (D-a)y = u & \rightarrow \quad y' - ay = u(t) \\ (D-a)u = g & \rightarrow \quad u' - au = g(t) \end{cases}$$

Solve the one for u first by using an integrating factor I.

$$I_1 = \exp\left[\int^t (-a)\, ds\right] = e^{-at}$$

Multiply both sides of the ODE for u by I.

$$e^{-at}u' - ae^{-at}u = g(t)e^{-at}$$

The left side can be written as $d/dt(Iu)$ by the product rule.

$$\frac{d}{dt}(e^{-at}u) = g(t)e^{-at}$$

Integrate both sides with respect to t.

$$e^{-at}u = \int^t g(s)e^{-as}\, ds + C_1$$

Multiply both sides by e^{at}.

$$u(t) = e^{at}\int^t g(s)e^{-as}\, ds + C_1 e^{at}$$

Substitute this result into the ODE for y.

$$y' - ay = e^{at}\int^t g(s)e^{-as}\, ds + C_1 e^{at}$$

Multiply both sides by I.

$$e^{-at}y' - ae^{-at}y = \int^t g(s)e^{-as}\, ds + C_1$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}(e^{-at}y) = \int^t g(s)e^{-as}\,ds + C_1$$

Integrate both sides with respect to t.

$$e^{-at}y = \int^t \int^q g(s)e^{-as}\,ds\,dq + C_1 t + C_2$$

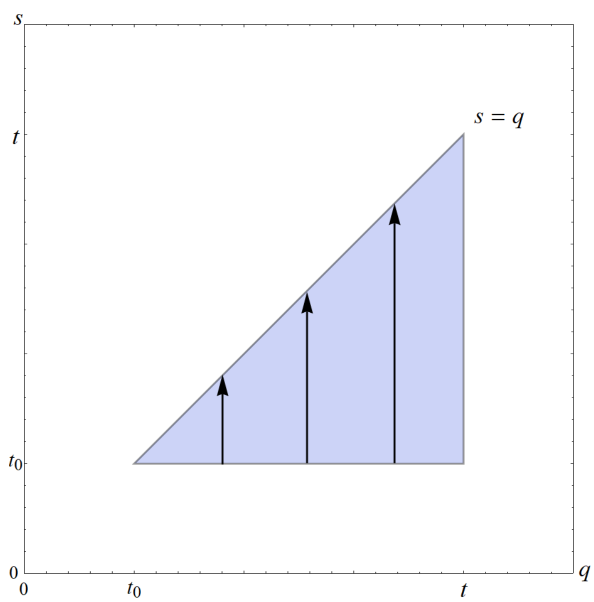
Multiply both sides by e^{at}.

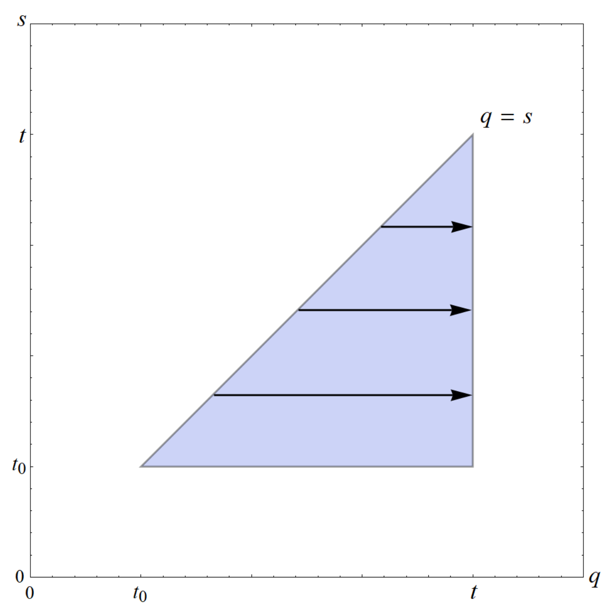
$$\begin{aligned} y(t) &= e^{at}\int^t \int^q g(s)e^{-as}\,ds\,dq + C_1 te^{at} + C_2 e^{at} \\ &= \int^t \int^q g(s)e^{a(t-s)}\,ds\,dq + C_1 te^{at} + C_2 e^{at} \end{aligned}$$

Since the initial conditions are given at $t = t_0$, the lower limits of integration will be set to t_0.

$$y(t) = \int_{t_0}^t \int_{t_0}^q g(s)e^{a(t-s)}\,ds\,dq + C_1 te^{at} + C_2 e^{at}$$

The current mode of integration in the qs-plane is shown below on the left.





Integrate over the domain as shown on the right to switch the order of integration.

$$\begin{aligned} y(t) &= \int_{t_0}^t \int_s^t g(s)e^{a(t-s)}\,dq\,ds + C_1 te^{at} + C_2 e^{at} \\ &= \int_{t_0}^t (q)\big|_s^t\, g(s)e^{a(t-s)}\,ds + C_1 te^{at} + C_2 e^{at} \\ &= \int_{t_0}^t (t-s)g(s)e^{a(t-s)}\,ds + C_1 te^{at} + C_2 e^{at} \end{aligned}$$

As a result, the general solution is

$$y(t) = \int_{t_0}^{t} (t-s)e^{a(t-s)} g(s)\, ds + C_1 t e^{at} + C_2 e^{at}.$$

Use the Leibnitz rule,

$$\frac{d}{dt}\int_{j(t)}^{h(t)} f(t,s)\, ds = \int_{j(t)}^{h(t)} \frac{\partial}{\partial t} f(t,s)\, ds + \frac{dh}{dt} f[t,h(t)] - \frac{dj}{dt} f[t,j(t)],$$

to differentiate the general solution.

$$\begin{aligned} y'(t) &= \int_{t_0}^{t} \frac{\partial}{\partial t}(t-s)e^{a(t-s)} g(s)\, ds + 1 \cdot (0)e^0 g(t) + C_1(e^{at} + ate^{at}) + aC_2 e^{at} \\ &= \int_{t_0}^{t} [e^{a(t-s)} + a(t-s)e^{a(t-s)}] g(s)\, ds + C_1(1+at)e^{at} + aC_2 e^{at} \end{aligned}$$

Apply the initial conditions now to determine C_1 and C_2.

$$\begin{aligned} y(t_0) &= C_1 t_0 e^{at_0} + C_2 e^{at_0} = 0 \\ y'(t_0) &= C_1(1+at_0)e^{at_0} + aC_2 e^{at_0} = 0 \end{aligned}$$

This system is only satisfied if $C_1 = 0$ and $C_2 = 0$. Therefore,

$$y(t) = \int_{t_0}^{t} (t-s)e^{a(t-s)} g(s)\, ds.$$

Method Using Variation of Parameters

$$L[y] = (D-a)^2 y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0$$

Distribute the operator to obtain a second-order ODE.

$$(D^2 - 2aD + a^2)y = g(t)$$

$$y'' - 2ay' + a^2 y = g(t)$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 2ay_c' + a^2 y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 2a(re^{rt}) + a^2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 2ar + a^2 = 0$$

$$(r-a)^2 = 0$$

$$r = \{a\}$$

One solution to equation (1) is then $y_c = e^{at}$. Use the method of reduction of order to obtain the general solution for $y_c(t)$: Plug $y_c(t) = c(t)e^{at}$ into equation (1) and then solve the resulting ODE for $c(t)$.

$$[c(t)e^{at}]'' - 2a[c(t)e^{at}]' + a^2[c(t)e^{at}] = 0$$

Evaluate the derivatives.

$$[c'(t)e^{at} + ac(t)e^{at}]' - 2a[c'(t)e^{at} + ac(t)e^{at}] + a^2[c(t)e^{at}] = 0$$

$$[c''(t)e^{at} + ac'(t)e^{at} + ac'(t)e^{at} + a^2 c(t)e^{at}] - 2a[c'(t)e^{at} + ac(t)e^{at}] + a^2[c(t)e^{at}] = 0$$

$$c''(t)e^{at} + \cancel{ac'(t)e^{at}} + \cancel{ac'(t)e^{at}} + \cancel{a^2 c(t)e^{at}} - \cancel{2ac'(t)e^{at}} - \cancel{2a^2 c(t)e^{at}} + \cancel{a^2 c(t)e^{at}} = 0$$

$$c''(t)e^{at} = 0$$

Divide both sides by e^{at}.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_3$$

Integrate both sides with respect to t once more.

$$c(t) = C_3 t + C_4$$

The complementary solution is then

$$\begin{aligned} y_c(t) &= c(t)e^{at} \\ &= (C_3 t + C_4)e^{at} \\ &= C_3 te^{at} + C_4 e^{at}. \end{aligned}$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_3(t)te^{at} + C_4(t)e^{at}$$

It satisfies the following ODE.

$$y_p'' - 2ay_p' + a^2 y_p = g(t)$$

Substitute the previous formula for $y_p(t)$.

$$[C_3(t)te^{at} + C_4(t)e^{at}]'' - 2a[C_3(t)te^{at} + C_4(t)e^{at}]' + a^2[C_3(t)te^{at} + C_4(t)e^{at}] = g(t)$$

Evaluate the derivatives.

$$\begin{aligned} &[C_3'(t)te^{at} + C_3(t)e^{at} + aC_3(t)te^{at} + C_4'(t)e^{at} + aC_4(t)e^{at}]' \\ &- 2a[C_3'(t)te^{at} + C_3(t)e^{at} + aC_3(t)te^{at} + C_4'(t)e^{at} + aC_4(t)e^{at}] + a^2[C_3(t)te^{at} + C_4(t)e^{at}] = g(t) \end{aligned}$$

$$\begin{aligned} &[C_3''(t)te^{at} + C_3'(t)e^{at} + aC_3'(t)te^{at} + C_3'(t)e^{at} + aC_3(t)e^{at} + aC_3'(t)te^{at} + aC_3(t)e^{at} + a^2C_3(t)te^{at} \\ &\qquad + C_4''(t)e^{at} + aC_4'(t)e^{at} + aC_4'(t)e^{at} + a^2C_4(t)e^{at}] \\ &- 2a[C_3'(t)te^{at} + C_3(t)e^{at} + aC_3(t)te^{at} + C_4'(t)e^{at} + aC_4(t)e^{at}] + a^2[C_3(t)te^{at} + C_4(t)e^{at}] = g(t) \end{aligned}$$

Simplify the left side.

$$te^{at}C_3''(t) + 2e^{at}C_3'(t) + e^{at}C_4''(t) = g(t)$$

If we set

$$te^{at}C_3''(t) + 2e^{at}C_3'(t) = 0, \tag{2}$$

then the previous equation reduces to

$$e^{at}C_4''(t) = g(t). \tag{3}$$

The aim now is to solve this system of two equations for $C_3(t)$ and $C_4(t)$. Start by dividing both sides of equation (2) by te^{at}.

$$\begin{aligned} C_3''(t) + \frac{2}{t}C_3'(t) &= 0 \\ \frac{C_3''(t)}{C_3'(t)} &= -\frac{2}{t} \end{aligned}$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dt}\ln C_3'(t) = -\frac{2}{t}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$\begin{aligned} \ln C_3'(t) &= -2\ln t \\ &= \ln t^{-2} \end{aligned}$$

Exponentiate both sides.

$$C_3'(t) = t^{-2}$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_3(t) = -t^{-1}$$

Divide both sides of equation (3) by e^{at}.

$$C_4''(t) = g(t)e^{-at}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_4'(t) = \int^t g(s)e^{-as}\,ds$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_4(t) = \int^t \int^q g(s)e^{-as}\,ds\,dq$$

The initial conditions are provided at $t = t_0$, so the arbitrary lower limits of integration will be set to t_0.

$$C_4(t) = \int_{t_0}^t \int_{t_0}^q g(s)e^{-as}\,ds\,dq$$

Switch the order of integration.

$$\begin{aligned} C_4(t) &= \int_{t_0}^t \int_s^t g(s)e^{-as}\,dq\,ds \\ &= \int_{t_0}^t (q)\big|_s^t\, g(s)e^{-as}\,ds \\ &= \int_{t_0}^t (t-s)g(s)e^{-as}\,ds \end{aligned}$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_3(t)te^{at} + C_4(t)e^{at} \\ &= [-t^{-1}]te^{at} + e^{at}\left[\int_{t_0}^t (t-s)g(s)e^{-as}\,ds\right] \\ &= -e^{at} + \int_{t_0}^t (t-s)g(s)e^{a(t-s)}\,ds, \end{aligned}$$

and consequently, the general solution is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_3te^{at} + C_4e^{at} - e^{at} + \int_{t_0}^t (t-s)g(s)e^{a(t-s)}\,ds \\ &= C_3te^{at} + (C_4 - 1)e^{at} + \int_{t_0}^t (t-s)g(s)e^{a(t-s)}\,ds \\ &= C_3te^{at} + C_5e^{at} + \int_{t_0}^t (t-s)g(s)e^{a(t-s)}\,ds. \end{aligned}$$

Differentiate the solution using the Leibnitz rule.

$$\begin{aligned} y'(t) &= C_3(e^{at} + ate^{at}) + aC_5e^{at} + \int_{t_0}^{t} \frac{\partial}{\partial t}(t-s)g(s)e^{a(t-s)}\,ds + 1\cdot(0)g(t)e^0 \\ &= C_3(1+at)e^{at} + aC_5e^{at} + \int_{t_0}^{t}[e^{a(t-s)} + a(t-s)e^{a(t-s)}]g(s)\,ds \end{aligned}$$

Apply the initial conditions now to determine C_3 and C_5.

$$\begin{aligned} y(t_0) &= C_3t_0e^{at_0} + C_5e^{at_0} = 0 \\ y'(t_0) &= C_3(1+at_0)e^{at_0} + aC_5e^{at_0} = 0 \end{aligned}$$

This system is satisfied only if $C_3 = 0$ and $C_5 = 0$. Therefore,

$$y(t) = \int_{t_0}^{t}(t-s)g(s)e^{a(t-s)}\,ds.$$