Problem 10

In each of Problems 5 through 12, find the general solution of the given differential equation. In Problems 11 and 12, \( g \) is an arbitrary continuous function.

\[
y'' - 2y' + y = e^t/(1 + t^2)
\]

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[
y(t) = y_c(t) + y_p(t)
\]

The complementary solution satisfies the associated homogeneous equation.

\[
y''_c - 2y'_c + y_c = 0 \tag{1}
\]

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[
y_c = e^{rt} \rightarrow y'_c = re^{rt} \rightarrow y''_c = r^2 e^{rt}
\]

Substitute these expressions into the ODE.

\[
r^2 e^{rt} - 2(re^{rt}) + e^{rt} = 0
\]

Divide both sides by \( e^{rt} \).

\[
r^2 - 2r + 1 = 0
\]

\[
(r - 1)^2 = 0
\]

\[
r = \{1\}
\]

One solution to equation (1) is \( y_c = e^t \). Use the method of reduction of order to obtain the general solution: Plug \( y_c(t) = c(t)e^t \) into equation (1) to obtain an ODE for \( c(t) \).

\[
[c(t)e^t]' - 2[c(t)e^t]' + [c(t)e^t] = 0
\]

Evaluate the derivatives.

\[
[c'(t)e^t + c(t)e^t]' - 2[c'(t)e^t + c(t)e^t]' + [c(t)e^t] = 0
\]

\[
[c''(t)e^t + c'(t)e^t + c'(t)e^t + c(t)e^t]' - 2[c'(t)e^t + c(t)e^t]' + [c(t)e^t] = 0
\]

\[
c''(t)e^t + c'(t)e^t + c'(t)e^t + c(t)e^t - 2c'(t)e^t - 2c(t)e^t + c(t)e^t = 0
\]

\[
c''(t)e^t = 0
\]

Integrate both sides with respect to \( t \).

\[
c'(t) = C_1
\]

Integrate both sides with respect to \( t \) once more.

\[
c(t) = C_1 t + C_2
\]
As a result,

\[ y_c(t) = c(t) e^t = (C_1 t + C_2) e^t = C_1 t e^t + C_2 e^t. \]

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in \( y_c(t) \) to vary.

\[ y_p(t) = C_1(t) t e^t + C_2(t) e^t \]

It satisfies the following ODE.

\[ y_p'' - 2y'_p + y_p = \frac{e^t}{1 + t^2} \]

Evaluate the derivatives.

\[
\begin{align*}
[C_1(t) t e^t + C_2(t) e^t]''' & - 2[C_1(t) t e^t + C_2(t) e^t]' + [C_1(t) t e^t + C_2(t) e^t] = \frac{e^t}{1 + t^2} \\
[C_1''(t) t e^t + C_1'(t) t e^t + C_1(t) e^t + C_2''(t) e^t + C_2'(t) e^t + C_2(t) e^t] & - 2[C_1'(t) t e^t + C_1(t) e^t + C_1(t) t e^t + C_2'(t) t e^t + C_2(t) e^t + C_2(t) e^t] \\
& + [C_1(t) t e^t + C_2(t) e^t] = \frac{e^t}{1 + t^2}
\end{align*}
\]

Simplify the left side.

\[
\begin{align*}
C_1''(t) t e^t + \frac{C_1'(t) t e^t + C_1(t) e^t}{C_1'(t) t e^t + C_1(t) e^t + C_1(t) t e^t + C_1(t) t e^t + C_1(t) t e^t + C_1(t) e^t + C_2'(t) t e^t + C_2''(t) e^t + C_2'(t) e^t + C_2(t) e^t + C_2(t) e^t + C_2(t) e^t} & - 2C_1'(t) t e^t - 2C_1(t) e^t - 2C_1'(t) t e^t - 2C_2'(t) e^t - 2C_2(t) e^t \\
& + C_1(t) t e^t + C_2(t) e^t = \frac{e^t}{1 + t^2}
\end{align*}
\]

Divide both sides by \( e^t \).

\[
C_1''(t) t + 2C_1'(t) + C_2''(t) = \frac{1}{1 + t^2}
\]

If we set

\[ C_2''(t) = 0, \tag{2} \]

then the previous equation reduces to

\[ C_1''(t) t + 2C_1'(t) = \frac{1}{1 + t^2} \tag{3} \]

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The aim now is to solve this system of equations. Solve equation (2) first for $C_2(t)$. Integrate both sides of it with respect to $t$, setting the integration constant to zero.

$$C'_2(t) = 0$$

Integrate both sides of it with respect to $t$ once more, setting the integration constant to zero.

$$C_2(t) = 0$$

Divide both sides of equation (3) by $t$.

$$C''_1(t) + \frac{2}{t}C'_1(t) = \frac{1}{t(1+t^2)}$$

Use an integrating factor $I_1$ to solve it.

$$I_1 = \exp\left(\int^t \frac{2}{s} \, ds\right) = e^{2\ln t} = e^{\ln t^2} = t^2$$

Multiply both sides of the previous equation by $I_1$.

$$t^2C''_1(t) + 2tC'_1(t) = \frac{t}{1+t^2}$$

The left side can be written as $d/dt[I_1C'_1(t)]$ by the product rule.

$$\frac{d}{dt}[t^2C'_1(t)] = \frac{t}{1+t^2}$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$t^2C'_1(t) = \frac{1}{2} \ln(1+t^2)$$

Divide both sides by $t^2$.

$$C'_1(t) = \frac{1}{2t^2} \ln(1+t^2)$$

Integrate both sides with respect to $t$ once more, setting the integration constant to zero.

$$C_1(t) = \int^t \frac{1}{2s^2} \ln(1+s^2) \, ds$$

$$= \frac{1}{2} \int^t \frac{d}{ds} \left(-\frac{1}{s}\right) \ln(1+s^2) \, ds$$

$$= \frac{1}{2} \left[ \left(-\frac{1}{s}\right) \ln(1+s^2) \right]^t \int^t \left(-\frac{1}{s}\right) \left(\frac{2s}{1+s^2}\right) \, ds$$

$$= \frac{1}{2} \left( \ln(1+t^2) - 2 \int^t \frac{ds}{1+s^2} \right)$$

$$= \frac{1}{2} \ln(1+t^2) + \frac{\tan^{-1} t}{t} + \frac{\ln \sqrt{1+t^2}}{t}$$

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Consequently, the particular solution is

\[
y_p(t) = C_1(t)te^t + C_2(t)e^t
= te^t \tan^{-1} t - e^t \ln \sqrt{1 + t^2}.
\]

Therefore,

\[
y(t) = y_c(t) + y_p(t)
= C_1te^t + C_2e^t + te^t \tan^{-1} t - e^t \ln \sqrt{1 + t^2}.
\]