Problem 13

In each of Problems 13 through 20, verify that the given functions $y_1$ and $y_2$ satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, $g$ is an arbitrary continuous function.

\[ t^2 y'' - 2y = 3t^2 - 1, \quad t > 0; \quad y_1(t) = t^2, \quad y_2(t) = t^{-1} \]

Solution

Verify that the first solution satisfies the associated homogeneous equation.

\[
\begin{align*}
    t^2 y''_1 - 2y_1 &= 0 \\
    t^2(t^2)'' - 2(t^2) &= 0 \\
    t^2(2) - 2(t^2) &= 0 \\
    0 &= 0
\end{align*}
\]

Now verify that the second solution satisfies the associated homogeneous equation.

\[
\begin{align*}
    t^2 y''_2 - 2y_2 &= 0 \\
    t^2 \left( \frac{1}{t^3} \right)'' - 2 \left( \frac{1}{t} \right) &= 0 \\
    t^2 \left( \frac{2}{t^3} \right) - 2 \left( \frac{1}{t} \right) &= 0 \\
    \frac{2}{t} - \frac{2}{t} &\neq 0 \\
    0 &= 0
\end{align*}
\]

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

\[ y(t) = y_c(t) + y_p(t) \]

By the principle of superposition, $y_c(t)$ is a linear combination of $y_1(t)$ and $y_2(t)$.

\[ y_c(t) = C_1 t^2 + C_2 t^{-1} \]

According to the method of variation of parameters, the particular solution is found by allowing the parameters in $y_c(t)$ to vary.

\[ y_p(t) = C_1(t) t^2 + C_2(t) t^{-1} \]

It satisfies the following ODE.

\[ t^2 y''_p - 2y_p = 3t^2 - 1 \]

Substitute the previous formula for $y_p(t)$.

\[ t^2[ C_1(t)t^2 + C_2(t)t^{-1} ]'' - 2[ C_1(t)t^2 + C_2(t)t^{-1} ] = 3t^2 - 1 \]
Evaluate the derivatives.

\[ t^2 [C'_1(t)t^2 + 2C_1(t)t + C'_2(t)t^{-1} - C_2(t)t^{-2}] - 2[C_1(t)t^2 + C_2(t)t^{-1}] = 3t^2 - 1 \]

\[ t^2[C''_1(t)t^2 + 2C'_1(t)t + 2C_1(t)t + C''_2(t)t^{-1} - C'_2(t)t^{-2} - C_2(t)t^{-2} + 2C(t)t^{-3}] \]

\[ - 2[C_1(t)t^2 + C_2(t)t^{-1}] = 3t^2 - 1 \]

\[ C''_1(t)t^4 + 2C'_1(t)t^3 + 2C_1(t)t^3 + 2C_1(t)t^2 + C''_2(t)t - C'_2(t) - C_2(t) + 2C(t)t^{-1} \]

\[ - 2C_1(t)t^2 - 2C_2(t)t^{-1} = 3t^2 - 1 \]

\[ C''_1(t)t^4 + 4C'_1(t)t^3 + C''_2(t)t - 2C'_2(t) = 3t^2 - 1 \]

If we set

\[ C''_2(t)t - 2C'_2(t) = 0, \quad (1) \]

then the previous equation reduces to

\[ C''_1(t)t^4 + 4C'_1(t)t^3 = 3t^2 - 1. \quad (2) \]

The aim now is to solve this system of equations for \( C_1(t) \) and \( C_2(t) \). Divide both sides of equation (1) by \( t \).

\[ C''_2(t) - \frac{2}{t}C'_2(t) = 0 \]

Use an integrating factor \( I_1 \) to solve it.

\[ I_1 = \exp \left( \int t^2 \frac{2}{t^2} \, ds \right) = e^{-2\ln t} = e^{\ln t^{-2}} = t^{-2} \]

Multiply both sides of the previous equation by \( I_1 \).

\[ \frac{1}{t^2} C''_2(t) - \frac{2}{t^3} C'_2(t) = 0 \]

The left side can be written as \( d/dt[I_1C'_2(t)] \) by the product rule.

\[ \frac{d}{dt} \left[ \frac{1}{t^2} C'_2(t) \right] = 0 \]

Integrate both sides with respect to \( t \), setting the integration constant to zero.

\[ \frac{1}{t^2} C'_2(t) = 0 \]

Multiply both sides by \( t^2 \).

\[ C'_2(t) = 0 \]

Integrate both sides with respect to \( t \) once more, setting the integration constant to zero.

\[ C_2(t) = 0 \]

Divide both sides of equation (2) by \( t^4 \).

\[ C''_1(t) + \frac{4}{t} C'_1(t) = \frac{3}{t^2} - \frac{1}{t^4} \]
Use an integrating factor $I_2$ to solve it.

$$I_2 = \exp \left( \int \frac{4}{s} \, ds \right) = e^{4 \ln t} = e^{\ln t^4} = t^4$$

Multiply both sides of the previous equation by $I_2$.

$$t^4 C''_1(t) + 4t^3 C'_1(t) = 3t^2 - 1$$

The left side can be written as $d/dt[I_2 C'_1(t)]$ by the product rule.

$$\frac{d}{dt}[t^4 C'_1(t)] = 3t^2 - 1$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$t^4 C'_1(t) = t^3 - t$$

Divide both sides by $t^4$.

$$C'_1(t) = \frac{1}{t^2}$$

Integrate both sides with respect to $t$ once more, setting the integration constant to zero.

$$C_1(t) = \ln t + \frac{1}{2t^2}$$

The particular solution is then

$$y_p(t) = C_1(t)y_1(t) + C_2(t)y_2(t)$$

$$= C_1(t)t^2 + C_2(t)t^{-1}$$

$$= \left( \ln t + \frac{1}{2t^2} \right) t^2$$

$$= t^2 \ln t + \frac{1}{2}.$$

Therefore, the general solution is

$$y(t) = y_c(t) + y_p(t)$$

$$= C_1 t^2 + C_2 t^{-1} + t^2 \ln t + \frac{1}{2}.$$