Problem 17

In each of Problems 13 through 20, verify that the given functions $y_1$ and $y_2$ satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, $g$ is an arbitrary continuous function.

\[ x^2y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0; \quad y_1(x) = x^2, \quad y_2(x) = x^2 \ln x \]

Solution

Verify that the first solution satisfies the associated homogeneous equation.

\[ x^2y_1'' - 3xy_1' + 4y_1 = 0 \]
\[ x^2(2) - 3x(2) + 4(2) = 0 \]
\[ 2x^2 - 6x^2 + 4x^2 = 0 \]
\[ 0 = 0 \]

Now verify that the second solution satisfies the associated homogeneous equation.

\[ x^2y_2'' - 3xy_2' + 4y_2 = 0 \]
\[ x^2(x^2 \ln x)'' - 3x(x^2 \ln x)' + 4(x^2 \ln x) = 0 \]
\[ x^2(2 \ln x + 2 + 1) - 3x(2x \ln x + x) + 4(x^2 \ln x) = 0 \]
\[ 2x^2 \ln x + 3x^2 - 6x^2 \ln x - 3x^2 + 4x^2 \ln x = 0 \]
\[ 0 = 0 \]

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(x)$ and the particular solution $y_p(x)$.

\[ y(x) = y_c(x) + y_p(x) \]

By the principle of superposition, $y_c(x)$ is a linear combination of $y_1(x)$ and $y_2(x)$.

\[ y_c(x) = C_1x^2 + C_2x^2 \ln x \]

According to the method of variation of parameters, the particular solution is found by allowing the parameters in $y_c(x)$ to vary.

\[ y_p(x) = C_1(x)x^2 + C_2(x)x^2 \ln x \]

It satisfies the following ODE.

\[ x^2y_p'' - 3xy_p' + 4y_p = x^2 \ln x \]
If we set 

\[ x^2[C_1(x)x^2 + C_2(x)x^2 \ln x]'' - 3x[C_1(x)x^2 + C_2(x)x^2 \ln x]' + 4[C_1(x)x^2 + C_2(x)x^2 \ln x] = x^2 \ln x \]

Evaluate the derivatives.

\[ x^2[C'_1(x)x^2 + 2C_1(x)x + C'_2(x)x^2 \ln x + 2C_2(x)x \ln x + C_2(x)x]' - 3x[C'_1(x)x^2 + 2C_1(x)x + C'_2(x)x^2 \ln x + 2C_2(x)x \ln x + C_2(x)x] + 4[C_1(x)x^2 + C_2(x)x^2 \ln x] = x^2 \ln x \]

\[ x^2[C''_1(x)x^2 + 2C'_1(x)x + 2C_1(x)x + C''_2(x)x^2 \ln x + 2C'_2(x)x \ln x + C_2(x)x + C'_2(x)x + C_2(x)x] - 3x[C'_1(x)x^2 + 2C_1(x)x + C'_2(x)x^2 \ln x + 2C_2(x)x \ln x + C_2(x)x] + 4[C_1(x)x^2 + C_2(x)x^2 \ln x] = x^2 \ln x \]

\[ x^4C'_1(x) + x^3C'_1(x) + (x^4 \ln x)C'_2(x) + (x^3 \ln x)C'_2(x) + 2x^3C'_2(x) = x^2 \ln x \]

If we set

\[ x^4C'_1(x) + x^3C'_1(x) = 0 \]  \hspace{1cm} (1)

then the previous equation reduces to

\[ (x^4 \ln x)C'_2(x) + (x^3 \ln x)C'_2(x) + 2x^3C'_2(x) = x^2 \ln x. \]  \hspace{1cm} (2)

The aim now is to solve this system of equations for \( C_1(x) \) and \( C_2(x) \). Divide both sides of equation (1) by \( x^4 \).

\[ C'_1(x) + \frac{1}{x}C'_1(x) = 0 \]

Use an integrating factor \( I_1 \) to solve it.

\[ I_1 = \exp \left( \int \frac{1}{s} \, ds \right) = e^{\ln x} = x \]

Multiply both sides of the previous equation by \( I_1 \).

\[ xC''_1(x) + C'_1(x) = 0 \]

The left side can be written as \( d/dx[I_1C'_1(x)] \) by the product rule.

\[ \frac{d}{dx}[xC'_1(x)] = 0 \]

Integrate both sides with respect to \( x \), setting the integration constant to zero.

\[ xC'_1(x) = 0 \]

Divide both sides by \( x \).

\[ C'_1(x) = 0 \]

Integrate both sides with respect to \( x \) once more, setting the integration constant to zero.

\[ C_1(x) = 0 \]
Factor the left side of equation (2)

\[(x^4 \ln x)C''_2(x) + (x^3 \ln x + 2x^3)C'_2(x) = x^2 \ln x\]

and then divide both sides by \(x^4 \ln x\).

\[C''_2(x) + \left(\frac{1}{x} + \frac{2}{x \ln x}\right) C'_2(x) = \frac{1}{x^2}\]

Use another integrating factor \(I_2\) to solve it.

\[I_2 = \exp \left[ \int \left( \frac{1}{s} + \frac{2}{s \ln s}\right) ds \right] = e^{\ln x + 2 \ln \ln x} = e^{\ln x} e^{\ln(\ln x)^2} = x(\ln x)^2\]

Multiply both sides of the previous equation by \(I_2\).

\[x(\ln x)^2 C''_2(x) + [(\ln x)^2 + 2 \ln x]C'_2(x) = \frac{(\ln x)^2}{x}\]

The left side can be written as \(d/dx[I_2 C'_2(x)]\) by the product rule.

\[\frac{d}{dx}[x(\ln x)^2 C'_2(x)] = \frac{(\ln x)^2}{x}\]

Integrate both sides with respect to \(x\), setting the integration constant to zero.

\[x(\ln x)^2 C'_2(x) = \frac{(\ln x)^3}{3}\]

Divide both sides by \(x(\ln x)^2\).

\[C'_2(x) = \frac{\ln x}{3x}\]

Integrate both sides with respect to \(x\) once more, setting the integration constant to zero.

\[C_2(x) = \frac{(\ln x)^2}{6}\]

The particular solution is then

\[y_p(x) = C_1(x) y_1(x) + C_2(x) y_2(x)\]

\[= C_1(x) x^2 + C_2(x) x^2 \ln x\]

\[= \left[ \frac{(\ln x)^2}{6} \right] x^2 \ln x\]

\[= \frac{x^2(\ln x)^3}{6}\].

Therefore, the general solution is

\[y(x) = y_c(x) + y_p(x)\]

\[= C_1 x^2 + C_2 x^2 \ln x + \frac{x^2(\ln x)^3}{6}\].

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