Problem 18

In each of Problems 13 through 20, verify that the given functions $y_1$ and $y_2$ satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, $g$ is an arbitrary continuous function.

$x^2 y'' + xy' + (x^2 - 0.25)y = 3x^{3/2} \sin x, \quad x > 0; \quad y_1(x) = x^{-1/2} \sin x, \quad y_2(x) = x^{-1/2} \cos x$

Solution

Verify that the first solution satisfies the associated homogeneous equation.

$x^2 y'' + xy' + (x^2 - 0.25)y = 0$

$x^2(x^{-1/2} \sin x)'' + x(x^{-1/2} \sin x)' + (x^2 - 0.25)(x^{-1/2} \sin x) = 0$

$x^2 \left( - \frac{1}{2} x^{-3/2} \sin x + x^{-1/2} \cos x \right)' + x \left( - \frac{1}{2} x^{-3/2} \sin x + x^{-1/2} \cos x \right) + (x^2 - 0.25)(x^{-1/2} \sin x) = 0$

$x^2 \left( \frac{3}{4} x^{-5/2} \sin x - \frac{1}{2} x^{-3/2} \cos x - \frac{1}{2} x^{-3/2} \cos x - x^{-1/2} \sin x \right) + x \left( - \frac{1}{2} x^{-3/2} \sin x + x^{-1/2} \cos x \right) + (x^2 - 0.25)(x^{-1/2} \sin x) = 0$

$0 = 0$

Now verify that the second solution satisfies the associated homogeneous equation.

$x^2 y'' + xy' + (x^2 - 0.25)y = 0$

$x^2(x^{-1/2} \cos x)'' + x(x^{-1/2} \cos x)' + (x^2 - 0.25)(x^{-1/2} \cos x) = 0$

$x^2 \left( - \frac{1}{2} x^{-3/2} \cos x - x^{-1/2} \sin x \right)' + x \left( - \frac{1}{2} x^{-3/2} \cos x - x^{-1/2} \sin x \right) + (x^2 - 0.25)(x^{-1/2} \cos x) = 0$

$x^2 \left( \frac{3}{4} x^{-5/2} \cos x + \frac{1}{2} x^{-3/2} \sin x + \frac{1}{2} x^{-3/2} \sin x - x^{-1/2} \cos x \right) + x \left( - \frac{1}{2} x^{-3/2} \cos x - x^{-1/2} \sin x \right) + (x^2 - 0.25)(x^{-1/2} \cos x) = 0$

$0 = 0$
0 = 0

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(x) \) and the particular solution \( y_p(x) \).

\[ y(x) = y_c(x) + y_p(x) \]

By the principle of superposition, \( y_c(x) \) is a linear combination of \( y_1(x) \) and \( y_2(x) \).

\[ y_c(x) = C_1 x^{-1/2} \sin x + C_2 x^{-1/2} \cos x \]

According to the method of variation of parameters, the particular solution is found by allowing the parameters in \( y_c(x) \) to vary.

\[ y_p(x) = C_1 x^{-1/2} \sin x + C_2(x) x^{-1/2} \cos x \]

It satisfies the following ODE.

\[ x^2 y'' + xy' + (x^2 - 0.25)y_p = 3x^{3/2} \sin x \]

Substitute the previous formula for \( y_p(x) \).

\[ x^2[C_1(x) x^{-1/2} \sin x + C_2(x) x^{-1/2} \cos x]' + x[C_1(x) x^{-1/2} \sin x + C_2(x) x^{-1/2} \cos x]' + (x^2 - 0.25)[C_1(x) x^{-1/2} \sin x + C_2(x) x^{-1/2} \cos x] = 3x^{3/2} \sin x \]

Evaluate the derivatives.

\[
x^2 \left[ C_1''(x)x^{-1/2} \sin x - \frac{1}{2} C_1'(x)x^{-3/2} \sin x + C_1'(x)x^{-1/2} \cos x + C_2'(x)x^{-1/2} \cos x \right] \\
- \frac{1}{2} C_2(x)x^{-3/2} \cos x - C_2(x)x^{-1/2} \sin x \right] \\
+ x \left[ C_1'(x)x^{-1/2} \sin x - \frac{1}{2} C_1(x)x^{-3/2} \sin x + C_1(x)x^{-1/2} \cos x + C_2'(x)x^{-1/2} \cos x \right] \\
- \frac{1}{2} C_2(x)x^{-3/2} \cos x - C_2(x)x^{-1/2} \sin x \right] \\
+ (x^2 - 0.25)[C_1(x)x^{-1/2} \sin x + C_2(x)x^{-1/2} \cos x] = 3x^{3/2} \sin x
\]

\[
x^2 \left[ C_1''(x)x^{-1/2} \sin x - \frac{1}{2} C_1'(x)x^{-3/2} \sin x + C_1'(x)x^{-1/2} \cos x - \frac{1}{2} C_1'(x)x^{-3/2} \sin x + \frac{3}{4} C_1(x)x^{-5/2} \sin x \right] \\
- \frac{1}{2} C_1(x)x^{-3/2} \cos x + C_1'(x)x^{-1/2} \cos x - \frac{1}{2} C_1(x)x^{-3/2} \cos x - C_1(x)x^{-1/2} \sin x \right] \\
+ C_2''(x)x^{-1/2} \cos x - \frac{1}{2} C_2'(x)x^{-3/2} \cos x - C_2'(x)x^{-1/2} \sin x - \frac{1}{2} C_2'(x)x^{-3/2} \cos x \right] \\
+ \frac{3}{4} C_2(x)x^{-5/2} \cos x + \frac{1}{2} C_2(x)x^{-3/2} \sin x - C_2'(x)x^{-1/2} \sin x + \frac{1}{2} C_2(x)x^{-3/2} \sin x - C_2(x)x^{-1/2} \cos x \right] \\
+ x \left[ C_1'(x)x^{-1/2} \sin x - \frac{1}{2} C_1(x)x^{-3/2} \sin x + C_1(x)x^{-1/2} \cos x + C_2'(x)x^{-1/2} \cos x \right] \\
- \frac{1}{2} C_2(x)x^{-3/2} \cos x - C_2(x)x^{-1/2} \sin x \right] \\
+ (x^2 - 0.25)[C_1(x)x^{-1/2} \sin x + C_2(x)x^{-1/2} \cos x] = 3x^{3/2} \sin x
\]
Simplify the left side.

\[ x^{3/2}(\sin x)C''_1(x) + 2x^{3/2}(\cos x)C'_1(x) + x^{3/2}(\cos x)C''_2(x) - 2x^{3/2}(\sin x)C'_2(x) = 3x^{3/2}\sin x \]

Divide both sides by \(x^{3/2} \).

\[ (\sin x)C''_1(x) + 2(\cos x)C'_1(x) + (\cos x)C''_2(x) - 2(\sin x)C'_2(x) = 3\sin x \]

If we set

\[ (\cos x)C''_2(x) - 2(\sin x)C'_2(x) = 0, \quad (1) \]

then the previous equation reduces to

\[ (\sin x)C''_1(x) + 2(\cos x)C'_1(x) = 3\sin x. \quad (2) \]

The aim now is to solve this system of equations for \(C_1(x)\) and \(C_2(x)\). Divide both sides of equation (1) by \(\cos x\).

\[ C''_2(x) - \frac{2\sin x}{\cos x}C'_2(x) = 0 \]

Use an integrating factor \(I_1\) to solve it.

\[ I_1 = \exp \left( \int \frac{-2\sin s}{\cos s} \, ds \right) = e^{2\ln \cos x} = e^{\ln \cos^2 x} = \cos^2 x \]

Multiply both sides of the previous equation by \(I_1\).

\[ (\cos^2 x)C''_2(x) - (2 \sin x \cos x)C'_2(x) = 0 \]

The left side can be written as \(d/dx[I_1C'_2(x)]\) by the product rule.

\[ \frac{d}{dx}[I_1C'_2(x)] = 0 \]

Integrate both sides with respect to \(x\), setting the integration constant to zero.

\[ (\cos^2 x)C'_2(x) = 0 \]

Divide both sides by \(\cos^2 x\).

\[ C'_2(x) = 0 \]

Integrate both sides with respect to \(x\) once more, setting the integration constant to zero.

\[ C_2(x) = 0 \]

Divide both sides of equation (2) by \(\sin x\).

\[ \frac{C''_1(x) + 2\cos x}{\sin x}C'_1(x) = 3 \]

Use another integrating factor \(I_2\) to solve it.

\[ I_2 = \exp \left( \int \frac{2\cos s}{\sin s} \, ds \right) = e^{2\ln \sin x} = e^{\ln \sin^2 x} = \sin^2 x \]

Multiply both sides of the previous equation by \(I_2\).

\[ (\sin^2 x)C''_1(x) + (2 \cos x \sin x)C'_1(x) = 3\sin^2 x \]

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The left side can be written as \( d/dx[I_2C'_1(x)] \) by the product rule.

\[
\frac{d}{dx}[(\sin^2 x)C'_1(x)] = \frac{3}{2}(1 - \cos 2x)
\]

Integrate both sides with respect to \( x \), setting the integration constant to zero.

\[
(\sin^2 x)C'_1(x) = \frac{3}{2}x - \frac{3}{4}\sin 2x = \frac{3}{2}x - \frac{3}{2}\sin x \cos x
\]

Divide both sides by \( \sin^2 x \).

\[
C'_1(x) = \frac{3}{2}x \csc^2 x - \frac{3}{2}\cot x
\]

Integrate both sides with respect to \( x \) once more, setting the integration constant to zero.

\[
C_1(x) = \int^x \left( \frac{3}{2}s \csc^2 s - \frac{3}{2}\cot s \right) ds
\]

\[
= \frac{3}{2} \int^x s \csc^2 s \, ds - \frac{3}{2} \int^x \cot s \, ds
\]

\[
= \frac{3}{2} \int^x s \frac{d}{ds}(-\cot s) \, ds - \frac{3}{2} \int^x \cot s \, ds
\]

\[
= \frac{3}{2} \left[ s(-\cot s) \bigg|_x^s - \int^x (1)(-\cot s) \, ds \right] - \frac{3}{2} \int^x \cot s \, ds
\]

\[
= \frac{3}{2} \left[ -x \cot x + \int^x \cot s \, ds \right] - \frac{3}{2} \int^x \cot s \, ds
\]

\[
= -\frac{3}{2}x \cot x
\]

The particular solution is then

\[
y_p(x) = C_1(x)y_1(x) + C_2(x)y_2(x)
\]

\[
= C_1(x)x^{-1/2}\sin x + C_2(x)x^{-1/2}\cos x
\]

\[
= \left( -\frac{3}{2}x \cot x \right) x^{-1/2}\sin x
\]

\[
= -\frac{3}{2}x^{1/2}\cos x.
\]

Therefore, the general solution is

\[
y(x) = y_c(x) + y_p(x)
\]

\[
= C_1(x)^{-1/2}\sin x + C_2(x)^{-1/2}\cos x - \frac{3}{2}x^{1/2}\cos x.
\]