Problem 24

Use the result of Problem 22 to find the solution of the initial value problem

\[ L[y] = (D - a)(D - b)y = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0, \]

where \( a \) and \( b \) are real numbers with \( a \neq b \).

Solution

Method Using Operator Factorization

Since the operator has been factored, the substitution \( u = (D - b)y \) is invited. The ODE then becomes

\[ (D - a)u = g(t). \]

As a result, the second-order ODE has been reduced to a system of (decoupled) first-order ODEs.

\[
\begin{align*}
(D - b)y &= u \quad \rightarrow \quad y' - by = u(t) \\
(D - a)u &= g \quad \rightarrow \quad u' - au = g(t)
\end{align*}
\]

Solve the one for \( u \) first by using an integrating factor \( I_1 \).

\[ I_1 = \exp \left[ \int (-a) \, ds \right] = e^{-at} \]

Multiply both sides of the ODE for \( u \) by \( I_1 \).

\[ e^{-at}u' - ae^{-at}u = g(t)e^{-at} \]

The left side can be written as \( d/dt (I_1u) \) by the product rule.

\[ \frac{d}{dt} (e^{-at}u) = g(t)e^{-at} \]

Integrate both sides with respect to \( t \).

\[ e^{-at}u = \int g(s)e^{-as} \, ds + C_1 \]

Multiply both sides by \( e^{at} \).

\[ u(t) = e^{at} \int g(s)e^{-as} \, ds + C_1e^{at} \]

Substitute this result into the ODE for \( y \).

\[ y' - by = e^{at} \int g(s)e^{-as} \, ds + C_1e^{at} \]

Use another integrating factor \( I_2 \) to solve it.

\[ I_2 = \exp \left[ \int (-b) \, ds \right] = e^{-bt} \]

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Multiply both sides of the previous equation by \( I_2 \).
\[
e^{-bt} y' - be^{-bt} y = e^{at} e^{-bt} \int g(s) e^{-as} \, ds + C_1 e^{at} e^{-bt}
\]
The left side can be written as \( d/dt (I_2y) \) by the product rule.
\[
\frac{d}{dt} (e^{-bt} y) = e^{(a-b)t} \int g(s) e^{-as} \, ds + C_1 e^{(a-b)t}
\]
Integrate both sides with respect to \( t \).
\[
e^{-bt} y = \int e^{(a-b)q} \int g(s) e^{-as} \, ds \, dq + \frac{C_1}{a-b} e^{(a-b)t} + C_2
\]
Multiply both sides by \( e^{bt} \) and use a new constant \( C_3 \) for \( C_1/(a-b) \).
\[
y(t) = e^{bt} \int e^{(a-b)q} \int g(s) e^{-as} \, ds \, dq + C_3 e^{at} + C_2 e^{bt}
\]
Since the initial conditions are given at \( t = t_0 \), the lower limits of integration will be set to \( t_0 \).
\[
y(t) = e^{bt} \int_{t_0}^{t} \int_{t_0}^{q} e^{(a-b)q} g(s) e^{-as} \, ds \, dq + C_3 e^{at} + C_2 e^{bt}
\]
The current mode of integration in the \( qs \)-plane is shown below on the left.

Integrate over the domain as shown on the right to switch the order of integration.
\[
y(t) = e^{bt} \int_{t_0}^{t} \int_{t_0}^{q} e^{(a-b)q} g(s) e^{-as} \, ds \, dq + C_3 e^{at} + C_2 e^{bt}
\]
\[
= e^{bt} \int_{t_0}^{t} \left( \frac{1}{a-b} e^{(a-b)q} \bigg|_{t_0}^{q} \right) g(s) e^{-as} \, ds + C_3 e^{at} + C_2 e^{bt}
\]
\[
= e^{bt} \frac{1}{a-b} \int_{t_0}^{t} \left[ e^{(a-b)q} - e^{(a-b)s} \right] g(s) e^{-as} \, ds + C_3 e^{at} + C_2 e^{bt}
\]
As a result, the general solution is
\[
y(t) = \frac{1}{a - b} \left[ e^{a(t-s)} - e^{b(t-s)} \right] g(s) \, ds + C_3 e^{at} + C_2 e^{bt}.
\]

Use the Leibnitz rule,
\[
d \left( \int_{h(t)}^{j(t)} f(t, s) \, ds \right) = \left. \frac{\partial}{\partial t} f(t, s) \right|_{h(t)}^{j(t)} - \frac{dh}{dt} f[t, h(t)] + \frac{dj}{dt} f[t, j(t)],
\]
to differentiate the general solution.
\[
y'(t) = \frac{1}{a - b} \int_{t_0}^{t} \frac{\partial}{\partial t} \left[ e^{a(t-s)} - e^{b(t-s)} \right] g(s) \, ds + 1 \cdot (e^a - e^b) g(t) + C_3 a e^{at} + C_2 b e^{bt}
\]
\[
= \frac{1}{a - b} \int_{t_0}^{t} \left[ a e^{a(t-s)} - b e^{b(t-s)} \right] g(s) \, ds + C_3 a e^{at} + C_2 b e^{bt}
\]

Apply the initial conditions now to determine \( C_2 \) and \( C_3 \).
\[
y(t_0) = C_3 e^{at_0} + C_2 e^{bt_0} = 0
\]
\[
y'(t_0) = C_3 a e^{at_0} + C_2 b e^{bt_0} = 0
\]

This system is only satisfied if \( C_2 = 0 \) and \( C_3 = 0 \). Therefore,
\[
y(t) = \frac{1}{a - b} \int_{t_0}^{t} \left[ e^{a(t-s)} - e^{b(t-s)} \right] g(s) \, ds.
\]
Method Using Variation of Parameters

\[ L[y] = (D - a)(D - b)y = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0 \]

Distribute the operator to obtain a second-order ODE.

\[ D(D - b)y - a(D - b)y = g(t) \]
\[ y'' - by' - ay' + aby = g(t) \]
\[ y'' - (a + b)y' + aby = g(t) \]

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y''_c - (a + b)y'_c + aby_c = 0 \]

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \quad \rightarrow \quad y'_c = re^{rt} \quad \rightarrow \quad y''_c = r^2 e^{rt} \]

Substitute these expressions into the ODE.

\[ r^2 e^{rt} - (a + b)(re^{rt}) + ab(e^{rt}) = g(t) \]

Divide both sides by \( e^{rt} \).

\[ r^2 - (a + b)r + ab = 0 \]

\[ r = \frac{(a + b) \pm \sqrt{(a + b)^2 - 4ab}}{2} = \frac{a + b \pm \sqrt{(a - b)^2}}{2} = \frac{a + b \pm (a - b)}{2} \]

\[ r = \{a, b\} \]

Two solutions to equation (1) are then \( y_c = e^{at} \) and \( y_c = e^{bt} \). By the principle of superposition, the general solution is a linear combination of these two.

\[ y_c(t) = C_4 e^{at} + C_5 e^{bt} \]

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in \( y_c(t) \) to vary.

\[ y_p(t) = C_4(t)e^{at} + C_5(t)e^{bt} \]

It satisfies the following ODE.

\[ y''_p - (a + b)y'_p + aby_p = g(t) \]

Substitute the previous formula for \( y_p(t) \).

\[ [C_4(t)e^{at} + C_5(t)e^{bt}]'' - (a + b)[C_4(t)e^{at} + C_5(t)e^{bt}]' + ab[C_4(t)e^{at} + C_5(t)e^{bt}] = g(t) \]

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Simplify the left side. 

\[ [C'_4(t)e^{at} + aC_4(t)e^{at} + C'_5(t)e^{bt} + bC_5(t)e^{bt}] 
- (a + b)[C'_4(t)e^{at} + aC_4(t)e^{at} + C'_5(t)e^{bt} + bC_5(t)e^{bt}] + ab[C_4(t)e^{at} + C_5(t)e^{bt}] = g(t) \]

\[ [C''_4(t)e^{at} + aC'_4(t)e^{at} + a^2C_4(t)e^{at} + aC'_5(t)e^{bt} + bC'_5(t)e^{bt} + bC'_5(t)e^{bt} + b^2C_5(t)e^{bt}] 
- (a + b)[C'_4(t)e^{at} + aC_4(t)e^{at} + C'_5(t)e^{bt} + bC_5(t)e^{bt}] + ab[C_4(t)e^{at} + C_5(t)e^{bt}] = g(t) \]

Integrate both sides with respect to \( t \), setting the integration constant to zero.

\[ e^{at}C''_4(t) + ae^{at}C'_4(t) - be^{at}C'_4(t) + e^{bt}C''_5(t) + be^{bt}C'_5(t) - ae^{bt}C'_5(t) = g(t) \]

Evaluate the derivatives.

Simplify the left side.

If we set

\[ e^{at}C'_4(t) + ae^{at}C'_4(t) + e^{bt}C'_5(t) + be^{bt}C'_5(t) = 0, \] (2)

then the previous equation reduces to

\[-be^{at}C'_4(t) - ae^{bt}C'_5(t) = g(t). \] (3)

The aim now is to solve this system of two equations for \( C_4(t) \) and \( C_5(t) \). Start by rewriting equation (2).

\[ \frac{d}{dt}[e^{at}C'_4(t)] + \frac{d}{dt}[e^{bt}C'_5(t)] = 0 \]

Integrate both sides with respect to \( t \), setting the integration constant to zero.

\[ e^{at}C'_4(t) + e^{bt}C'_5(t) = 0 \]

Solve for \( C'_4(t) \).

\[ C'_4(t) = -\frac{e^{bt}}{e^{at}} C'_5(t) \] (4)

Plug this formula into equation (3).

\[-be^{at} \left[ -\frac{e^{bt}}{e^{at}} C'_5(t) \right] - ae^{bt}C'_5(t) = g(t) \]

\[ C'_5(t)(be^{bt} - ae^{bt}) = g(t) \]

Divide both sides by \( be^{bt} - ae^{bt} \).

\[ C'_5(t) = \frac{g(t)e^{-bt}}{b - a} \]

Integrate both sides with respect to \( t \), setting the integration constant to zero.

\[ C_5(t) = \int^t \frac{g(s)e^{-bs}}{b - a} ds \]

Substitute the previous formula for \( C'_5(t) \) into equation (4).

\[ C'_4(t) = -\frac{e^{bt}}{e^{at}} \left[ \frac{g(t)e^{-bt}}{b - a} \right] \]

\[ = -\frac{g(t)e^{-at}}{b - a} \]

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Integrate both sides with respect to $t$, setting the integration constant to zero.

$$C_4(t) = - \int^t g(s)e^{-as} \, ds$$

The particular solution is then

$$y_p(t) = C_4(t)e^{at} + C_5(t)e^{bt}$$

$$= e^{at} \left[ - \int^t g(s)e^{-as} \, ds \right] + e^{bt} \left[ \int^t g(s)e^{-bs} \, ds \right]$$

$$= -e^{at} \int^t g(s)e^{-as} \, ds + e^{bt} \int^t g(s)e^{-bs} \, ds$$

$$= \int^t \left[ - \frac{g(s)e^{at-as}}{b-a} + \frac{g(s)e^{bt-bs}}{b-a} \right] \, ds$$

$$= \frac{1}{b-a} \int^t \left[ e^{b(t-s)} - e^{a(t-s)} \right] g(s) \, ds$$

$$= \frac{1}{b-a} \int_{t_0}^t \left[ e^{b(t-s)} - e^{a(t-s)} \right] g(s) \, ds$$

The lower limit of integration is arbitrary and has been set to $t_0$ because that's when the initial conditions are given. Consequently, the general solution is

$$y(t) = y_c(t) + y_p(t)$$

$$= C_4e^{at} + C_5e^{bt} + \frac{1}{b-a} \int_{t_0}^t \left[ e^{b(t-s)} - e^{a(t-s)} \right] g(s) \, ds.$$  

Differentiate it with respect to $t$ using the Leibnitz rule.

$$y'(t) = aC_4e^{at} + bC_5e^{bt} + \frac{1}{b-a} \int_{t_0}^t \left[ \frac{\partial}{\partial t} \left[ e^{b(t-s)} - e^{a(t-s)} \right] g(s) \, ds + 1 \cdot (e^0 - e^0)g(t)$$

$$= aC_4e^{at} + bC_5e^{bt} + \frac{1}{b-a} \int_{t_0}^t \left[ be^{b(t-s)} - ae^{a(t-s)} \right] g(s) \, ds$$

Apply the initial conditions now to determine $C_4$ and $C_5$.

$$y(t_0) = C_4e^{at_0} + C_5e^{bt_0} = 0$$

$$y'(t_0) = aC_4e^{at_0} + bC_5e^{bt_0} = 0$$

This system is only satisfied if $C_4 = 0$ and $C_5 = 0$. Therefore,

$$y(t) = \frac{1}{b-a} \int_{t_0}^t \left[ e^{b(t-s)} - e^{a(t-s)} \right] g(s) \, ds.$$