Problem 26

Use the result of Problem 22 to find the solution of the initial value problem

\[ L[y] = (D - a)^2 y = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0, \]

where \( a \) is any real number.

Solution

**Method Using Operator Factorization**

Since the operator has been factored, the substitution \( u = (D - a)y \) is invited. The ODE then becomes

\[ (D - a)u = g(t). \]

As a result, the second-order ODE has been reduced to a system of (decoupled) first-order ODEs.

\[
\begin{align*}
(D - a)y &= u \quad \rightarrow \quad y' - ay = u(t) \\
(D - a)u &= g \quad \rightarrow \quad u' - au = g(t)
\end{align*}
\]

Solve the one for \( u \) first by using an integrating factor \( I \).

\[
I_1 = \exp \left[ \int t (-a) \, ds \right] = e^{-at}
\]

Multiply both sides of the ODE for \( u \) by \( I \).

\[
e^{-at}u' - ae^{-at}u = g(t)e^{-at}
\]

The left side can be written as \( d/dt (Iu) \) by the product rule.

\[
\frac{d}{dt} (e^{-at}u) = g(t)e^{-at}
\]

Integrate both sides with respect to \( t \).

\[
e^{-at}u = \int g(s)e^{-as} \, ds + C_1
\]

Multiply both sides by \( e^{at} \).

\[
u(t) = e^{at} \int g(s)e^{-as} \, ds + C_1e^{at}
\]

Substitute this result into the ODE for \( y \).

\[
y' - ay = e^{at} \int g(s)e^{-as} \, ds + C_1e^{at}
\]

Multiply both sides by \( I \).

\[
e^{-at}y' - ae^{-at}y = \int g(s)e^{-as} \, ds + C_1
\]

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The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}(e^{-at}y) = \int_{t_0}^{t} g(s)e^{-as} \, ds + C_1$$

Integrate both sides with respect to $t$.

$$e^{-at}y = \int_{t_0}^{t} \int_{s}^{q} g(s)e^{-as} \, ds \, dq + C_1 t + C_2$$

Multiply both sides by $e^{at}$.

$$y(t) = e^{at} \int_{t_0}^{t} \int_{s}^{q} g(s)e^{a(t-s)} \, ds \, dq + C_1 te^{at} + C_2 e^{at}$$

Since the initial conditions are given at $t = t_0$, the lower limits of integration will be set to $t_0$.

$$y(t) = \int_{t_0}^{t} \int_{t_0}^{q} g(s)e^{a(t-s)} \, ds \, dq + C_1 te^{at} + C_2 e^{at}$$

The current mode of integration in the $qs$-plane is shown below on the left.

Integrate over the domain as shown on the right to switch the order of integration.

$$y(t) = \int_{t_0}^{t} \int_{s}^{t} g(s)e^{a(t-s)} \, dq \, ds + C_1 te^{at} + C_2 e^{at}$$

$$= \int_{t_0}^{t} (q)_{s}^{t} g(s)e^{a(t-s)} \, ds + C_1 te^{at} + C_2 e^{at}$$

$$= \int_{t_0}^{t} (t - s)g(s)e^{a(t-s)} \, ds + C_1 te^{at} + C_2 e^{at}$$

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As a result, the general solution is

$$y(t) = \int_{t_0}^{t} (t - s)e^{a(t-s)}g(s) \, ds + C_1te^{at} + C_2e^{at}.$$ 

Use the Leibnitz rule,

$$\frac{d}{dt} \int_{j(t)}^{h(t)} f(t, s) \, ds = \int_{j(t)}^{h(t)} \frac{\partial}{\partial t} f(t, s) \, ds + \frac{dh}{dt} f[t, h(t)] - \frac{d}{dt} f[t, j(t)],$$

to differentiate the general solution.

$$y'(t) = \int_{t_0}^{t} \frac{\partial}{\partial t} (t - s)e^{a(t-s)}g(s) \, ds + 1 \cdot (0)e^{0}g(t) + C_1(e^{at} + ate^{at}) + aC_2e^{at}$$

$$= \int_{t_0}^{t} [e^{a(t-s)} + a(t - s)e^{a(t-s)}]g(s) \, ds + C_1(1 + at)e^{at} + aC_2e^{at}$$

Apply the initial conditions now to determine $C_1$ and $C_2$.

$$y(t_0) = C_1t_0e^{at_0} + C_2e^{at_0} = 0$$

$$y'(t_0) = C_1(1 + at_0)e^{at_0} + aC_2e^{at_0} = 0$$

This system is only satisfied if $C_1 = 0$ and $C_2 = 0$. Therefore,

$$y(t) = \int_{t_0}^{t} (t - s)e^{a(t-s)}g(s) \, ds.$$
Method Using Variation of Parameters

\[ L[y] = (D - a)^2 y = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0 \]

Distribute the operator to obtain a second-order ODE.

\[ (D^2 - 2aD + a^2)y = g(t) \]
\[ y'' - 2ay' + a^2y = g(t) \]

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y''_c - 2ay'_c + a^2y_c = 0 \tag{1} \]

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \quad \rightarrow \quad y'_c = re^{rt} \quad \rightarrow \quad y''_c = r^2e^{rt} \]

Substitute these expressions into the ODE.

\[ r^2e^{rt} - 2a(re^{rt}) + a^2(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 - 2ar + a^2 = 0 \]
\[ (r - a)^2 = 0 \]
\[ r = \{a\} \]

One solution to equation (1) is then \( y_c = e^{at} \). Use the method of reduction of order to obtain the general solution for \( y_c(t) \): Plug \( y_c(t) = c(t)e^{at} \) into equation (1) and then solve the resulting ODE for \( c(t) \).

\[ [c(t)e^{at}]'' - 2a[c(t)e^{at}]' + a^2[c(t)e^{at}] = 0 \]

Evaluate the derivatives.

\[ [c'(t)e^{at} + ac(t)e^{at}]' - 2a[c'(t)e^{at} + ac(t)e^{at}] + a^2[c(t)e^{at}] = 0 \]
\[ [c''(t)e^{at} + ac'(t)e^{at} + ac(t)e^{at} + a^2c(t)e^{at}] - 2a[c'(t)e^{at} + ac(t)e^{at}] + a^2[c(t)e^{at}] = 0 \]
\[ c''(t)e^{at} + ac'(t)e^{at} + ac(t)e^{at} + a^2c(t)e^{at} - 2ac'(t)e^{at} - 2a^2c(t)e^{at} + a^2c(t)e^{at} = 0 \]
\[ c''(t)e^{at} = 0 \]

Divide both sides by \( e^{at} \).

\[ c''(t) = 0 \]

Integrate both sides with respect to \( t \).

\[ c'(t) = C_3 \]

Integrate both sides with respect to \( t \) once more.

\[ c(t) = C_3t + C_4 \]

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The complementary solution is then
\[ y_c(t) = c(t)e^{at} = (C_3t + C_4)e^{at} = C_3te^{at} + C_4e^{at}. \]

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in \( y_c(t) \) to vary.

\[ y_p(t) = C_3(t)te^{at} + C_4(t)e^{at} \]

It satisfies the following ODE.
\[ y'' - 2ay' + a^2y = g(t) \]

Substitute the previous formula for \( y_p(t) \).
\[ [C_3(t)te^{at} + C_4(t)e^{at}]'' - 2a[C_3(t)te^{at} + C_4(t)e^{at}]' + a^2[C_3(t)te^{at} + C_4(t)e^{at}] = g(t) \]

Evaluate the derivatives.
\[ [C_3'(t)te^{at} + C_3(t)e^{at} + aC_3(t)te^{at} + C_4'(t)e^{at} + aC_4(t)e^{at}]' \]
\[ - 2a[C_3'(t)te^{at} + C_3(t)e^{at} + aC_3(t)te^{at} + C_4'(t)e^{at} + aC_4(t)e^{at}] + a^2[C_3(t)te^{at} + C_4(t)e^{at}] = g(t) \]

Simplify the left side.
\[ te^{at}C_3''(t) + 2e^{at}C_3'(t) + e^{at}C_4'(t) = g(t) \]

If we set
\[ te^{at}C_3'(t) + 2e^{at}C_3'(t) = 0, \]
then the previous equation reduces to
\[ e^{at}C_4''(t) = g(t). \]

The aim now is to solve this system of two equations for \( C_3(t) \) and \( C_4(t) \). Start by dividing both sides of equation (2) by \( te^{at} \).
\[ C_3''(t) + \frac{2}{t}C_3'(t) = 0 \]
\[ C_3''(t) = \frac{2}{t}C_3'(t) \]

The left side can be written as the derivative of a logarithm by the chain rule.
\[ \frac{d}{dt} \ln C_3'(t) = -\frac{2}{t} \]

Integrate both sides with respect to \( t \), setting the integration constant to zero.
\[ \ln C_3'(t) = -2\ln t = \ln t^{-2} \]
Exponentiate both sides.

\[ C'_3(t) = t^{-2} \]

Integrate both sides with respect to \( t \) once more, setting the integration constant to zero.

\[ C_3(t) = -t^{-1} \]

Divide both sides of equation (3) by \( e^{at} \).

\[ C'_4(t) = g(t)e^{-at} \]

Integrate both sides with respect to \( t \), setting the integration constant to zero.

\[ C_4(t) = \int_{t_0}^{t} g(s)e^{-as} \, ds \]

Integrate both sides with respect to \( t \) once more, setting the integration constant to zero.

\[ C_4(t) = \int_{t_0}^{t} \int_{t_0}^{q} g(s)e^{-as} \, ds \, dq \]

The initial conditions are provided at \( t = t_0 \), so the arbitrary lower limits of integration will be set to \( t_0 \).

\[ C_4(t) = \int_{t_0}^{t} \int_{t_0}^{q} g(s)e^{-as} \, ds \, dq \]

Switch the order of integration.

\[ C_4(t) = \int_{t_0}^{t} \int_{t_0}^{t} g(s)e^{-as} \, dq \, ds \]

\[ = \int_{t_0}^{t} (q)|_{t_0}^{t} g(s)e^{-as} \, ds \]

\[ = \int_{t_0}^{t} (t-s)g(s)e^{-as} \, ds \]

The particular solution is then

\[ y_p(t) = C_3(t)te^{at} + C_4(t)e^{at} \]

\[ = [-t^{-1}]te^{at} + e^{at} \left[ \int_{t_0}^{t} (t-s)g(s)e^{-as} \, ds \right] \]

\[ = -e^{at} + \int_{t_0}^{t} (t-s)g(s)e^{a(t-s)} \, ds, \]

and consequently, the general solution is

\[ y(t) = y_c(t) + y_p(t) \]

\[ = C_3te^{at} + C_4e^{at} - e^{at} + \int_{t_0}^{t} (t-s)g(s)e^{a(t-s)} \, ds \]

\[ = C_3te^{at} + (C_4 - 1)e^{at} + \int_{t_0}^{t} (t-s)g(s)e^{a(t-s)} \, ds \]

\[ = C_3te^{at} + C_5e^{at} + \int_{t_0}^{t} (t-s)g(s)e^{a(t-s)} \, ds. \]
Differentiate the solution using the Leibnitz rule.

\[ y'(t) = C_3(e^{at} + ate^{at}) + aC_5e^{at} + \int_{t_0}^{t} \frac{\partial}{\partial t}(t-s)g(s)e^{a(t-s)} \, ds + 1 \cdot (0)g(t)e^{0} \]
\[ = C_3(1 + at)e^{at} + aC_5e^{at} + \int_{t_0}^{t} [e^{a(t-s)} + a(t - s)e^{a(t-s)}]g(s) \, ds \]

Apply the initial conditions now to determine \( C_3 \) and \( C_5 \).

\[ y(t_0) = C_3t_0e^{a(t_0)} + C_5e^{a(t_0)} = 0 \]
\[ y'(t_0) = C_3(1 + at_0)e^{a(t_0)} + aC_5e^{a(t_0)} = 0 \]

This system is satisfied only if \( C_3 = 0 \) and \( C_5 = 0 \). Therefore,

\[ y(t) = \int_{t_0}^{t} (t - s)g(s)e^{a(t-s)} \, ds. \]