Problem 32

In each of Problems 29 through 32, use the method outlined in Problem 28 to solve the given differential equation.

\[(1 - t) y'' + ty' - y = 2(t - 1)^2 e^{-t}, \quad 0 < t < 1; \quad y_1(t) = e^t \quad (\text{see Problem 16})\]

Solution

Since one solution is known, the method of reduction of order can be applied to determine the general solution. Substitute \(y(t) = c(t)e^t\) into the ODE and solve the resulting ODE for \(c(t)\).

\[(1 - t)[c(t)e^t]'' + t[c(t)e^t]' - [c(t)e^t] = 2(t - 1)^2 e^{-t}\]

Evaluate the derivatives.

\[(1 - t)[c'(t)e^t + c(t)e^t]' + t[c'(t)e^t + c(t)e^t] - [c(t)e^t] = 2(t - 1)^2 e^{-t}\]

Simplify the left side.

\[(1 - t)e^t c''(t) + (2 - t)e^t c'(t) = 2(1 - t)^2 e^{-t}\]

Divide both sides by \((1 - t)e^t\).

\[c''(t) + \frac{2 - t}{1 - t} c'(t) = 2(1 - t)e^{-2t}\]

Use an integrating factor \(I\) to solve this ODE.

\[I = \exp \left( \int t \frac{2 - s}{1 - s} ds \right) = \exp \left( 2 \int \frac{t}{1 - s} ds - \int \frac{s}{1 - s} ds \right) = e^{-2\ln(1-t)+t+\ln(1-t)} = (1 - t)^{-1} e^t\]

Multiply both sides of the previous equation by \(I\).

\[
\frac{e^t}{1 - t} c''(t) + \frac{2 - t}{(1 - t)^2} e^t c'(t) = 2e^{-t}
\]

The left side can be written as \(d/dt[IC'(t)]\) by the product rule.

\[
\frac{d}{dt} \left[ \frac{e^t}{1 - t} c'(t) \right] = 2e^{-t}
\]

Integrate both sides with respect to \(t\).

\[
\frac{e^t}{1 - t} c'(t) = -2e^{-t} + C_1
\]

Divide both sides by \(I\).

\[c'(t) = -2(1 - t)e^{-2t} + C_1(1 - t)e^{-t}\]

Integrate both sides with respect to \(t\) once more.

\[c(t) = \frac{1}{2} (1 - 2t)e^{-2t} + C_1 t e^{-t} + C_2\]
Therefore,

\[ y(t) = c(t)e^t \]

\[ = \left[ \frac{1}{2}(1 - 2t)e^{-2t} + C_1te^{-t} + C_2 \right] e^t \]

\[ = \frac{1}{2}(1 - 2t)e^{-t} + C_1 t + C_2 e^t. \]

The terms containing \( C_1 \) and \( C_2 \) are the second and first solutions, respectively, to the associated homogeneous equation, and the first term is the particular solution.