Problem 7

In each of Problems 5 through 12, find the general solution of the given differential equation. In Problems 11 and 12, \( g \) is an arbitrary continuous function.

\[
y'' + 4y' + 4y = t - 2e^{-2t}, \quad t > 0
\]

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[
y(t) = y_c(t) + y_p(t)
\]

The complementary solution satisfies the associated homogeneous equation.

\[
y''_c + 4y'_c + 4y_c = 0 \quad (1)
\]

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[
y_c = e^{rt} \rightarrow y'_c = re^{rt} \rightarrow y''_c = r^2e^{rt}
\]

Substitute these expressions into the ODE.

\[
r^2e^{rt} + 4(re^{rt}) + 4(e^{rt}) = 0
\]

Divide both sides by \( e^{rt} \).

\[
r^2 + 4r + 4 = 0
\]

\[
(r + 2)^2 = 0
\]

\[
r = \{-2\}
\]

One solution to equation (1) is \( y_c = e^{-2t} \). Use the method of reduction of order to obtain the general solution: Plug \( y_c(t) = c(t)e^{-2t} \) into equation (1) to obtain an ODE for \( c(t) \).

\[
[c(t)e^{-2t}]'' + 4[c(t)e^{-2t}]' + 4[c(t)e^{-2t}] = 0
\]

Evaluate the derivatives.

\[
[c'(t)e^{-2t} - 2c(t)e^{-2t}]' + 4[c'(t)e^{-2t} - 2c(t)e^{-2t}] + 4[c(t)e^{-2t}] = 0
\]

\[
[c''(t)e^{-2t} - 2c'(t)e^{-2t} - 2c'(t)e^{-2t} + 4c(t)e^{-2t}] + 4[c'(t)e^{-2t} - 2c(t)e^{-2t}] + 4[c(t)e^{-2t}] = 0
\]

\[
c''(t)e^{-2t} - 2c'(t)e^{-2t} - 2c'(t)e^{-2t} + 4c(t)e^{-2t} + 4c'(t)e^{-2t} - 8c(t)e^{-2t} + 4c(t)e^{-2t} = 0
\]

\[
c''(t)e^{-2t} = 0
\]

Multiply both sides by \( e^{2t} \).

\[
c''(t) = 0
\]

Integrate both sides with respect to \( t \).

\[
c'(t) = C_1
\]

Integrate both sides with respect to \( t \) once more.

\[
c(t) = C_1t + C_2
\]
As a result,

\[ y_c(t) = c(t)e^{-2t} \]
\[ = (C_1t + C_2)e^{-2t} \]
\[ = C_1te^{-2t} + C_2e^{-2t}. \]

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in \( y_c(t) \) to vary.

\[ y_p(t) = C_1(t)te^{-2t} + C_2(t)e^{-2t} \]

It satisfies the following ODE.

\[ y_p'' + 4y_p' + 4y_p = t^{-2}e^{-2t} \]

Substitute the previous formula in for \( y_p(t) \).

\[ [C_1(t)te^{-2t} + C_2(t)e^{-2t}]'' + 4[C_1(t)te^{-2t} + C_2(t)e^{-2t}]' + 4[C_1(t)te^{-2t} + C_2(t)e^{-2t}] = t^{-2}e^{-2t} \]

Evaluate the derivatives.

\[ [C_1'(t)te^{-2t} + C_1(t)e^{-2t} - 2C_1(t)te^{-2t} + C_2'(t)e^{-2t} - 2C_2(t)e^{-2t}]' \]
\[ + 4[C_1'(t)te^{-2t} + C_1(t)e^{-2t} - 2C_1(t)te^{-2t} + C_2'(t)e^{-2t} - 2C_2(t)e^{-2t}] + 4[C_1(t)te^{-2t} + C_2(t)e^{-2t}] = t^{-2}e^{-2t} \]

\[ [C_1''(t)te^{-2t} + C_1'(t)e^{-2t} - 2C_1'(t)te^{-2t} + C_1'(t)e^{-2t} - 2C_1'(t)te^{-2t} - 2C_1(t)e^{-2t} + 4C_1(t)te^{-2t} + C_2'(t)e^{-2t} - 2C_2'(t)e^{-2t} + 4C_2(t)e^{-2t}] \]
\[ + 4[C_1'(t)te^{-2t} + C_1(t)e^{-2t} - 2C_1(t)te^{-2t} + C_2'(t)e^{-2t} - 2C_2(t)e^{-2t}] + 4[C_1(t)te^{-2t} + C_2(t)e^{-2t}] = t^{-2}e^{-2t} \]

Simplify the left side.

\[ C_1''(t)te^{-2t} + C_1'(t)e^{-2t} - 2C_1'(t)te^{-2t} + C_1'(t)e^{-2t} - 2C_1'(t)te^{-2t} - 2C_1(t)e^{-2t} + 4C_1(t)te^{-2t} + C_2''(t)e^{-2t} - 2C_2'(t)e^{-2t} + 4C_2(t)e^{-2t} + 4C_1(t)te^{-2t} + 4C_2(t)e^{-2t} = t^{-2}e^{-2t} \]

\[ C_1''(t)te^{-2t} + 2C_1'(t)te^{-2t} + C_2''(t)e^{-2t} = t^{-2}e^{-2t} \]

Multiply both sides by \( e^{2t} \).

\[ C_1''(t)t + 2C_1'(t) + C_2''(t) = t^{-2} \]

If we set

\[ C_1''(t)t + 2C_1'(t) = 0, \]

then the previous equation reduces to

\[ C_2''(t) = t^{-2} \]
The aim now is to solve this system of equations. Solve equation (2) first for $C_1(t)$. Divide both sides of it by $t$.

$$C''_1(t) + \frac{2}{t} C'_1(t) = 0$$

Use an integrating factor $I_1$ to solve it.

$$I_1 = \exp \left( \int \frac{2}{s} \, ds \right) = e^{2\ln t} = e^\ln t^2 = t^2$$

Multiply both sides of the previous equation by $I_1$.

$$t^2 C''_1(t) + 2t C'_1(t) = 0$$

The left side can be written as $d/dt[I_1 C'_1(t)]$ by the product rule.

$$\frac{d}{dt} [t^2 C'_1(t)] = 0$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$t^2 C'_1(t) = 0$$

Divide both sides by $t^2$.

$$C'_1(t) = 0$$

Integrate both sides with respect to $t$ once more, setting the integration constant to zero.

$$C_1(t) = 0$$

Now solve equation (3) for $C_2(t)$. Integrate both sides of it with respect to $t$, setting the integration constant to zero.

$$C'_2(t) = -t^{-1}$$

Integrate both sides of it with respect to $t$ once more, setting the integration constant to zero.

$$C_2(t) = -\ln t$$

Consequently, the particular solution is

$$y_p(t) = C_1(t)te^{-2t} + C_2(t)e^{-2t}$$

$$= -e^{-2t} \ln t.$$ 

Therefore,

$$y(t) = y_c(t) + y_p(t)$$

$$= C_1 te^{-2t} + C_2 e^{-2t} - e^{-2t} \ln t.$$