Problem 8

In each of Problems 5 through 12, find the general solution of the given differential equation. In Problems 11 and 12, \( g \) is an arbitrary continuous function.

\[
y'' + 4y = 3 \csc 2t, \quad 0 < t < \pi/2
\]

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution \( y_c(t) \) and the particular solution \( y_p(t) \).

\[
y(t) = y_c(t) + y_p(t)
\]

The complementary solution satisfies the associated homogeneous equation.

\[
y'' + 4y = 0 \quad (1)
\]

This is a homogeneous ODE with constant coefficients, so the solution is of the form \( y_c = e^{rt} \).

\[
y_c = e^{rt} \quad \rightarrow \quad y'_c = re^{rt} \quad \rightarrow \quad y''_c = r^2 e^{rt}
\]

Substitute these expressions into the ODE.

\[
r^2 e^{rt} + 4(e^{rt}) = 0
\]

Divide both sides by \( e^{rt} \).

\[
r^2 + 4 = 0
\]

\[
r = \{-2i, 2i\}
\]

Two solutions to equation (1) are then \( y_c = e^{-2it} \) and \( y_c = e^{2it} \). By the principle of superposition, the general solution is a linear combination of these two.

\[
y_c(t) = C_1 e^{-2it} + C_2 e^{2it}
\]

\[
= C_1 [\cos(-2t) + i \sin(-2t)] + C_2 [\cos(2t) + i \sin(2t)]
\]

\[
= C_1 [\cos(2t) - i \sin(2t)] + C_2 [\cos(2t) + i \sin(2t)]
\]

\[
= C_1 \cos 2t - iC_1 \sin 2t + C_2 \cos 2t + iC_2 \sin 2t
\]

\[
= (C_1 + C_2) \cos 2t + (-iC_1 + iC_2) \sin 2t
\]

\[
= C_3 \cos 2t + C_4 \sin 2t
\]

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in \( y_c(t) \) to vary.

\[
y_p(t) = C_3(t) \cos 2t + C_4(t) \sin 2t
\]

It satisfies the following ODE.

\[
y'' + 4y = 3 \csc 2t
\]

Substitute the previous formula in for \( y_p(t) \).

\[
[C_3(t) \cos 2t + C_4(t) \sin 2t]' + 4[C_3(t) \cos 2t + C_4(t) \sin 2t] = 3 \csc 2t
\]

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Evaluate the derivatives.

\[ C_3'(t) \cos 2t - 2C_3(t) \sin 2t + C_4'(t) \sin 2t + 2C_4(t) \cos 2t]' + 4[C_3(t) \cos 2t + C_4(t) \sin 2t] = 3 \csc 2t \]

\[ C_3''(t) \cos 2t - 2C_3'(t) \sin 2t + 2C_3(t) \sin 2t - 4C_3(t) \cos 2t - 4C_4(t) \cos 2t - 4C_4'(t) \cos 2t - 4C_4(t) \sin 2t = 3 \csc 2t \]

If we set

\[ C_3''(t) \cos 2t - 4C_3'(t) \sin 2t = 0, \tag{2} \]

then the previous equation reduces to

\[ C_4''(t) \sin 2t + 4C_4'(t) \cos 2t = 3 \csc 2t. \tag{3} \]

The aim now is to solve this system of two equations for \( C_3(t) \) and \( C_4(t) \). Start by dividing equation (2) by \( \cos 2t \).

\[ C_3''(t) - 4 \frac{\sin 2t}{\cos 2t} C_3'(t) = 0 \]

Use an integrating factor \( I_1 \) to solve it.

\[ I_1 = \exp \left( \int t \left( -4 \frac{\sin 2s}{\cos 2s} ds \right) \right) = e^{2 \ln \cos 2t} = e^{\ln \cos^2 2t} = \cos^2 2t \]

Multiply both sides of the previous equation by \( I_1 \).

\[ (\cos^2 2t)C_3''(t) - (4 \sin 2t \cos 2t)C_3'(t) = 0 \]

The left side can be written as \( d/dt[I_1 C_3'(t)] \) by the product rule.

\[ \frac{d}{dt}[I_1 C_3'(t)] = 0 \]

Integrate both sides with respect to \( t \), setting the integration constant to zero.

\[ (\cos^2 2t)C_3'(t) = 0 \]

Divide both sides by \( \cos^2 2t \).

\[ C_3'(t) = 0 \]

Integrate both sides with respect to \( t \) once more, setting the integration constant to zero.

\[ C_3(t) = 0 \]

Divide both sides of equation (3) by \( \sin 2t \).

\[ C_4''(t) + 4 \frac{\cos 2t}{\sin 2t} C_4'(t) = 3 \csc^2 2t \]

Use an integrating factor \( I_2 \) to solve it.

\[ I_2 = \exp \left( \int t \left( 4 \frac{\cos 2s}{\sin 2s} ds \right) \right) = e^{2 \ln \sin 2t} = e^{\ln \sin^2 2t} = \sin^2 2t \]
Multiply both sides of the previous equation by $I_2$.

$$(\sin^2 2t)C_4''(t) + (4 \cos 2t \sin 2t)C_4'(t) = 3.$$ 

The left side can be written as $d/dt[I_2C_4'(t)]$ by the product rule.

$$\frac{d}{dt}[(\sin^2 2t)C_4'(t)] = 3$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$(\sin^2 2t)C_4'(t) = 3t$$

Divide both sides by $\sin^2 2t$.

$$C_4'(t) = 3t \csc^2 2t$$

Integrate both sides with respect to $t$ once more, setting the integration constant to zero.

$$C_4(t) = \int^t 3s \csc^2 2s \, ds$$

Make the following substitution.

$$u = 2s \quad \rightarrow \quad \frac{u}{2} = s$$

$$du = 2 \, ds \quad \rightarrow \quad \frac{du}{2} = ds$$

As a result,

$$C_4(t) = \int^{2t} 3 \frac{u}{2} \csc^2 u \, \frac{du}{2}$$

$$= \frac{3}{4} \int^{2t} u \csc^2 u \, du$$

$$= \frac{3}{4} \int^{2t} \frac{du}{du} (- \cot u) \, du$$

$$= \frac{3}{4} \left[ u (- \cot u) \right]^{2t}_1 - \int^{2t} (- \cot u) \, du$$

$$= \frac{3}{4} \left[ -2t \cot 2t + \int^{2t} \cot u \, du \right]$$

$$= \frac{3}{4} (-2t \cot 2t + \ln |\sin 2t|).$$

Since $0 < t < \pi/2$, the absolute value sign can be removed. The particular solution is then

$$y_p(t) = C_3(t) \cos 2t + C_4(t) \sin 2t$$

$$= \frac{3}{4} [-2t \cos 2t + (\sin 2t) \ln \sin 2t].$$

Therefore,

$$y(t) = y_c(t) + y_p(t)$$

$$= C_3 \cos 2t + C_4 \sin 2t + \frac{3}{4} [-2t \cos 2t + (\sin 2t) \ln \sin 2t].$$

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