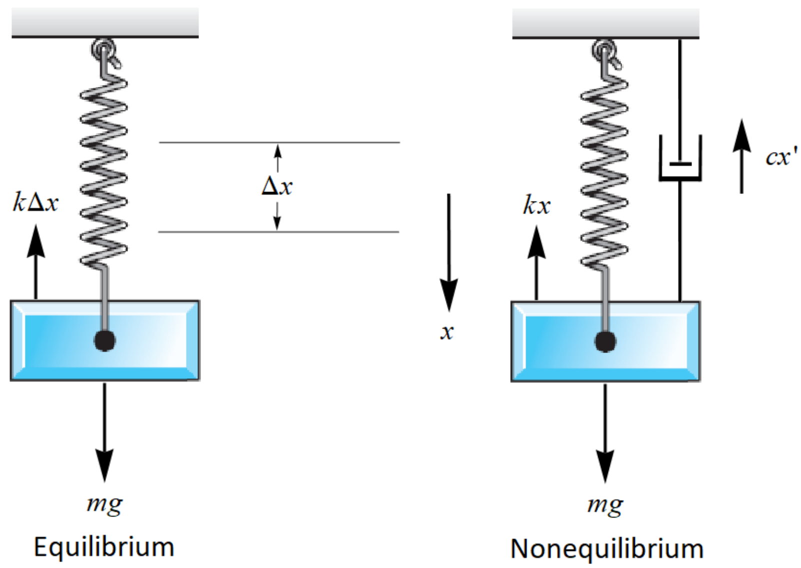
Problem 9

A mass of 20 g stretches a spring 5 cm. Suppose that the mass is also attached to a viscous damper with a damping constant of 400 dyn · s/cm. If the mass is pulled down an additional 2 cm and then released, find its position u at any time t. Plot u versus t. Determine the quasi frequency and the quasi period. Determine the ratio of the quasi period to the period of the corresponding undamped motion. Also find the time τ such that $|u(t)| < 0.05$ cm for all $t > \tau$.

Solution

Start by drawing a free-body diagram of the mass. The three forces acting on it are due to the spring, the damper (dashpot), and gravity.



If the mass is stationary and hanging from the spring, the gravitational and spring forces balance each other.

$$mg = k\Delta x$$

From this equation, k can be determined.

$$\left(20 \text{ g} \times \frac{1 \text{ kg}}{1000 \text{ g}}\right)\left(9.81 \frac{\text{m}}{\text{s}^2}\right) = k\left(5 \text{ cm} \times \frac{1 \text{ m}}{100 \text{ cm}}\right)$$

$$0.1962 \text{ N} = 0.05k \text{ m}$$

$$k = 3.924 \frac{\text{N}}{\text{m}}$$

Now we will apply Newton's second law in the x-direction to obtain the equation of motion for the mass.

$$\sum F_x = ma_x$$

$$-cx' - kx + mg = ma_x$$

Use the fact that acceleration is the second derivative of position with respect to time.

$$-cx' - kx + mg = mx''$$

$$mx'' + cx' + kx = mg$$

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$x(t) = x_c(t) + x_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$mx_c'' + cx_c' + kx_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $x_c = e^{rt}$.

$$x_c = e^{rt} \quad \rightarrow \quad x_c' = re^{rt} \quad \rightarrow \quad x_c'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$m(r^2 e^{rt}) + c(re^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + cr + k = 0$$

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

For this problem

$$\begin{aligned} c &= 400\ \cancel{\text{dyn}} \cdot \frac{\text{s}}{\cancel{\text{cm}}} \times \frac{1\ \text{N}}{10^5\ \cancel{\text{dyn}}} \times \frac{100\ \cancel{\text{cm}}}{1\ \text{m}} = 0.4\ \text{N} \cdot \frac{\text{s}}{\text{m}} \\ m &= 0.02\ \text{kg}, \end{aligned}$$

which means $c^2 - 4mk < 0$.

$$\begin{aligned} r &= \left\{ \frac{-c - i\sqrt{4mk - c^2}}{2m}, \frac{-c + i\sqrt{4mk - c^2}}{2m} \right\} \\ &= \left\{ -\frac{c}{2m} - i\mu, -\frac{c}{2m} + i\mu \right\} \end{aligned}$$

Two solutions to equation (1) are $x_c = e^{(-c/2m - i\mu)t}$ and $x_c = e^{(-c/2m + i\mu)t}$. By the principle of superposition, the general solution for x_c is a linear combination of the two.

$$\begin{aligned} x_c(t) &= C_1 e^{(-c/2m - i\mu)t} + C_2 e^{(-c/2m + i\mu)t} \\ &= C_1 e^{-ct/2m - i\mu t} + C_2 e^{-ct/2m + i\mu t} \\ &= C_1 e^{-ct/2m} e^{-i\mu t} + C_2 e^{-ct/2m} e^{i\mu t} \\ &= C_1 e^{-ct/2m} [\cos(-\mu t) + i \sin(-\mu t)] + C_2 e^{-ct/2m} [\cos(\mu t) + i \sin(\mu t)] \\ &= C_1 e^{-ct/2m} [\cos(\mu t) - i \sin(\mu t)] + C_2 e^{-ct/2m} [\cos(\mu t) + i \sin(\mu t)] \\ &= C_1 e^{-ct/2m} \cos \mu t - iC_1 e^{-ct/2m} \sin \mu t + C_2 e^{-ct/2m} \cos \mu t + iC_2 e^{-ct/2m} \sin \mu t \\ &= (C_1 + C_2) e^{-ct/2m} \cos \mu t + (-iC_1 + iC_2) e^{-ct/2m} \sin \mu t \\ &= C_3 e^{-ct/2m} \cos \mu t + C_4 e^{-ct/2m} \sin \mu t \end{aligned}$$

On the other hand, the particular solution satisfies

$$mx_p'' + cx_p' + kx_p = mg.$$

Because the inhomogeneous term is a constant, the particular solution is a constant as well: $x_p(t) = A$. Substitute this into the equation to determine A.

$$m(A)'' + c(A)' + k(A) = mg$$

$$kA = mg$$

$$A = \frac{mg}{k}$$

So then $x_p(t) = mg/k$, which means that the general solution for x is

$$x(t) = C_3 e^{-ct/2m} \cos \mu t + C_4 e^{-ct/2m} \sin \mu t + \frac{mg}{k}.$$

Take a derivative of it with respect to t.

$$x'(t) = -C_3 \frac{c}{2m} e^{-ct/2m} \cos \mu t - C_3 \mu e^{-ct/2m} \sin \mu t - C_4 \frac{c}{2m} e^{-ct/2m} \sin \mu t + C_4 \mu e^{-ct/2m} \cos \mu t$$

Now apply the initial conditions,

$$x(0) = 5 \text{ cm} + 2 \text{ cm} = 7 \cancel{\text{cm}} \times \frac{1 \text{ m}}{100 \cancel{\text{cm}}} = 0.07 \text{ m}$$

$$x'(0) = 0 \ \frac{\text{m}}{\text{s}},$$

to determine C_3 and C_4.

$$x(0) = C_3 + \frac{mg}{k} = 0.07$$

$$x'(0) = -C_3 \frac{c}{2m} + C_4 \mu = 0$$

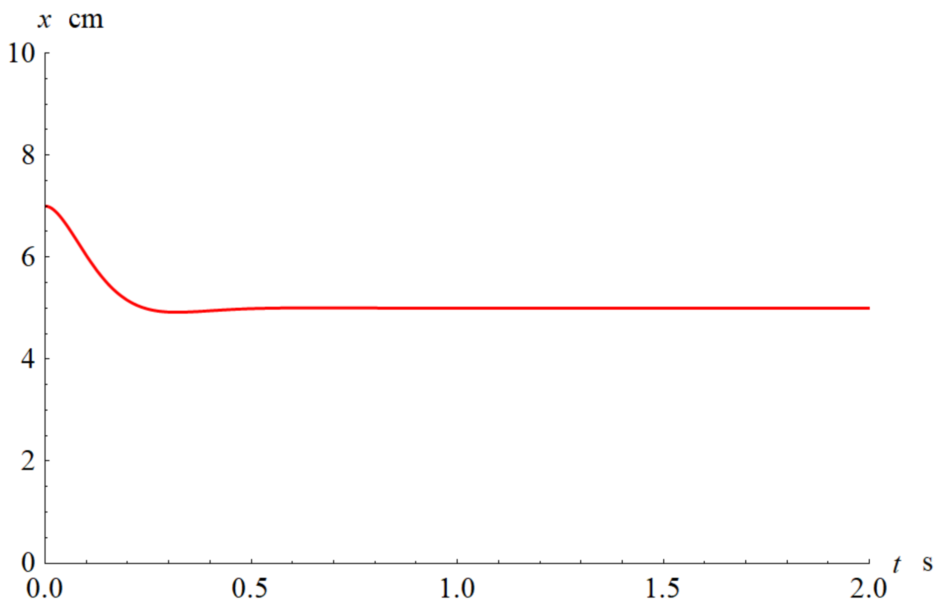
Solving this system of equations yields

$$C_3 = 0.07 - \frac{mg}{k} \quad \text{and} \quad C_4 = \frac{c}{2m\mu} \left(0.07 - \frac{mg}{k}\right).$$

Therefore,

$$\begin{aligned} x(t) &= \left(0.07 - \frac{mg}{k}\right) e^{-ct/2m} \cos \mu t + \frac{c}{2m\mu} \left(0.07 - \frac{mg}{k}\right) e^{-ct/2m} \sin \mu t + \frac{mg}{k} \\ &= \left(0.07 - \frac{mg}{k}\right) e^{-10t} \cos \frac{\sqrt{4mk - c^2}}{2m} t + \frac{c}{\sqrt{4mk - c^2}} \left(0.07 - \frac{mg}{k}\right) e^{-10t} \sin \frac{\sqrt{4mk - c^2}}{2m} t + \frac{mg}{k}. \end{aligned}$$

As $x(t)$ is in meters, multiply the result by 100 to convert it to centimeters.



The quasi angular frequency is

$$\mu = \frac{\sqrt{4mk - c^2}}{2m} \approx 9.81 \ \frac{\text{rad}}{\text{s}},$$

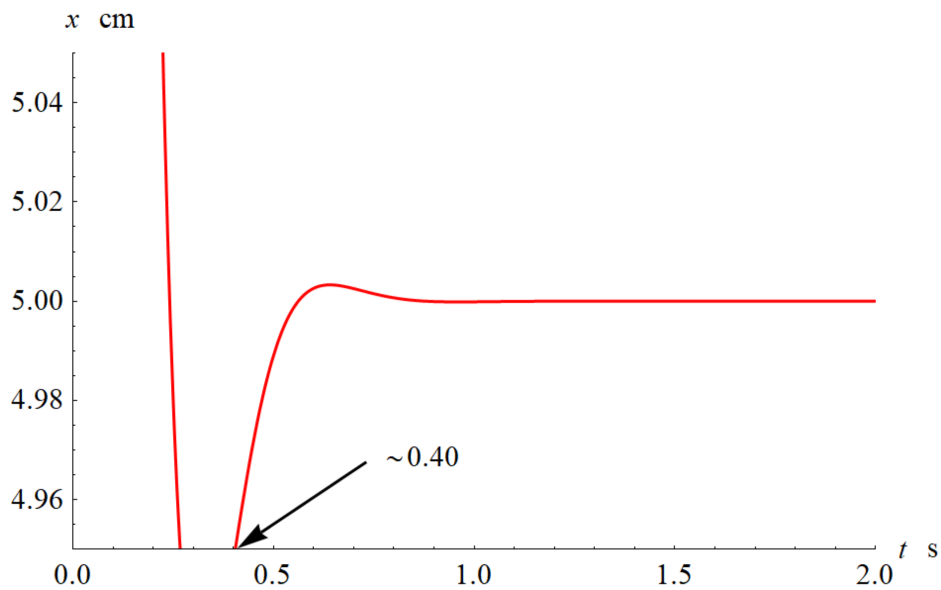
and the quasi period is

$$\begin{aligned} T_d &= \frac{2\pi}{\mu} \\ &= \frac{4\pi m}{\sqrt{4mk - c^2}} \\ &\approx 0.641 \text{ s.} \end{aligned}$$

The ratio of the quasi period to the period of the corresponding undamped motion is

$$\begin{aligned} \frac{T_d}{T} &= \frac{\frac{2\pi}{\mu}}{\frac{2\pi}{\omega}} \\ &= \frac{\omega}{\mu} \\ &= \frac{\sqrt{\frac{k}{m}}}{\frac{\sqrt{4mk - c^2}}{2m}} \\ &\approx 1.43. \end{aligned}$$

Zoom in the graph to within 0.05 centimeters of the equilibrium position.



We see that for $t \gtrsim 0.40$ s, the amplitude of x is within 0.05 centimeters of the equilibrium position for all t.