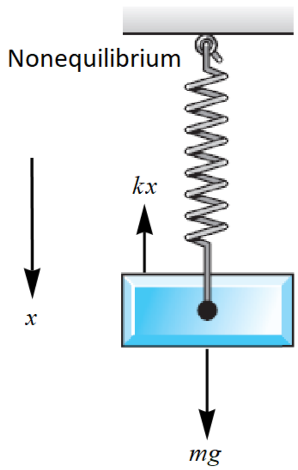
Problem 14

Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$, where L is the elongation of the spring due to the mass, and g is the acceleration due to gravity.

Solution

Start by drawing a free-body diagram for the mass hanging from a spring.



Apply Newton's second law to obtain the equation of motion.

$$\sum F_x = ma_x$$

The only two forces are due to the spring and gravity.

$$-kx + mg = ma_x$$

Use the fact that acceleration is the second derivative of position.

$$-kx + mg = mx''$$

$$mx'' + kx = mg$$

This is a linear inhomogeneous ODE, so the general solution can be written as a sum of the complementary solution and the particular solution.

$$x(t) = x_c(t) + x_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$mx_c'' + kx_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $x_c = e^{rt}$.

$$x_c = e^{rt} \quad \rightarrow \quad x_c' = re^{rt} \quad \rightarrow \quad x_c'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$m(r^2e^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + k = 0$$

$$r^2 = -\frac{k}{m}$$

$$r = \left\{-i\sqrt{\frac{k}{m}}, i\sqrt{\frac{k}{m}}\right\} = \{-i\omega, i\omega\}$$

Two solutions to equation (1) are then $x_c = e^{-i\omega t}$ and $x_c = e^{i\omega t}$. By the principle of superposition, the general solution for $x_c(t)$ is a linear combination of these two.

$$\begin{aligned} x_c(t) &= C_1e^{-i\omega t} + C_2e^{i\omega t} \\ &= C_1[\cos(-\omega t) + i\sin(-\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\ &= C_1[\cos(\omega t) - i\sin(\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\ &= C_1\cos\omega t - iC_1\sin\omega t + C_2\cos\omega t + iC_2\sin\omega t \\ &= (C_1 + C_2)\cos\omega t + (-iC_1 + iC_2)\sin\omega t \\ &= C_3\cos\omega t + C_4\sin\omega t \end{aligned}$$

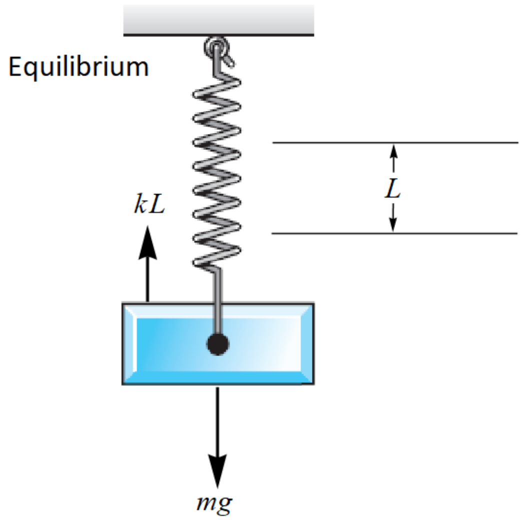
Since C_3 and C_4 are arbitrary, we can introduce an amplitude A and a phase δ in order to write the two sinusoidal terms as one.

$$\begin{aligned} x_c(t) &= A\cos\delta\cos\omega t + A\sin\delta\sin\omega t \\ &= A\cos(\omega t - \delta) \\ &= A\cos\left(\sqrt{\frac{k}{m}}t - \delta\right) \end{aligned}$$

The period of the motion is

$$T = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}}.$$

In order to write this in terms of the elongation L and g, we have to consider the spring in equilibrium.



Here the force due to gravity balances the force due to the spring.

$$mg = kL$$

Solve this for m/k.

$$\frac{m}{k} = \frac{L}{g}$$

Therefore,

$$T = 2\pi\sqrt{\frac{L}{g}}.$$