# Problem 19

Assume that the system described by the equation $mu'' + \gamma u' + ku = 0$ is either critically damped or overdamped. Show that the mass can pass through the equilibrium position at most once, regardless of the initial conditions.
*Hint:* Determine all possible values of $t$ for which $u = 0$.

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**Solution**

## Case 1: Critical Damping

Suppose first that the motion is critically damped. Then the ratio of $\gamma^2$ to $4km$ is 1, and $\gamma = \sqrt{4km}$. As a result, the ODE becomes

$$mu'' + \sqrt{4km}u' + ku = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $u = e^{rt}$.

$$u = e^{rt} \quad \rightarrow \quad u' = re^{rt} \quad \rightarrow \quad u'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for $r$.

$$m(r^2 e^{rt}) + \sqrt{4km}(re^{rt}) + k(e^{rt}) = 0$$

Divide both sides by $e^{rt}$.

$$mr^2 + \sqrt{4km}r + k = 0$$

$$r = \frac{-\sqrt{4km} \pm \sqrt{4km - 4km}}{2m} = -\frac{\sqrt{4km}}{2m} = -\sqrt{\frac{k}{m}}$$

One solution to equation (1) is $u = e^{-\sqrt{k/mt}}$. Apply the method of reduction of order to obtain the general solution: Plug in $u(t) = c(t)e^{-\sqrt{k/mt}}$ to equation (1) and solve the resulting ODE for $c(t)$.

$$m[c(t)e^{-\sqrt{k/mt}}]'' + \sqrt{4km}[c(t)e^{-\sqrt{k/mt}}]' + k[c(t)e^{-\sqrt{k/mt}}] = 0$$

Evaluate the derivatives.

$$m\left[c'(t)e^{-\sqrt{k/mt}} - c(t)\sqrt{\frac{k}{m}}e^{-\sqrt{k/mt}}\right]' + \sqrt{4km}\left[c'(t)e^{-\sqrt{k/mt}} - c(t)\sqrt{\frac{k}{m}}e^{-\sqrt{k/mt}}\right] + k[c(t)e^{-\sqrt{k/mt}}] = 0$$

$$m\left[c''(t)e^{-\sqrt{k/mt}} - \cancel{c'(t)\sqrt{\frac{k}{m}}e^{-\sqrt{k/mt}}} - \cancel{c'(t)\sqrt{\frac{k}{m}}e^{-\sqrt{k/mt}}} + \cancel{c(t)\frac{k}{m}e^{-\sqrt{k/mt}}}\right] + \sqrt{4km}\left[\cancel{c'(t)e^{-\sqrt{k/mt}}} - \cancel{c(t)\sqrt{\frac{k}{m}}e^{-\sqrt{k/mt}}}\right] + \cancel{k[c(t)e^{-\sqrt{k/mt}}]} = 0$$

$$mc''(t)e^{-\sqrt{k/mt}} = 0$$

Multiply both sides by $e^{\sqrt{k/mt}}/m$.

$$c''(t) = 0$$

Integrate both sides with respect to $t$.

$$c'(t) = C_1$$

Integrate both sides with respect to $t$ once more.

$$c(t) = C_1 t + C_2$$

The general solution to equation (1) is then

$$\begin{aligned} u(t) &= c(t)e^{-\sqrt{k/m}t} \\ &= (C_1 t + C_2)e^{-\sqrt{k/m}t} \\ &= C_1 t e^{-\sqrt{k/m}t} + C_2 e^{-\sqrt{k/m}t}. \end{aligned}$$

Take a derivative of it with respect to $t$.

$$u'(t) = C_1 e^{-\sqrt{k/m}t} - C_1\sqrt{\frac{k}{m}}te^{-\sqrt{k/m}t} - C_2\sqrt{\frac{k}{m}}e^{-\sqrt{k/m}t}$$

Apply the initial conditions, $u(0) = u_0$ and $u'(0) = v_0$, to determine $C_1$ and $C_2$.

$$\begin{aligned} u(0) &= C_2 = u_0 \\ u'(0) &= C_1 - C_2\sqrt{\frac{k}{m}} = v_0 \end{aligned}$$

Solving this system of equations for $C_1$ and $C_2$ yields

$$C_1 = v_0 + \sqrt{\frac{k}{m}}u_0 \quad \text{and} \quad C_2 = u_0.$$

Thus, the amplitude of the critically damped system is

$$u(t) = \left(v_0 + \sqrt{\frac{k}{m}}u_0\right)te^{-\sqrt{k/m}t} + u_0 e^{-\sqrt{k/m}t}.$$

Set $u(t) = 0$ and solve for $t$ to determine the time at which the mass passes through the equilibrium position.

$$\begin{aligned} 0 &= \left(v_0 + \sqrt{\frac{k}{m}}u_0\right)te^{-\sqrt{k/m}t} + u_0 e^{-\sqrt{k/m}t} \\ 0 &= \left(v_0 + \sqrt{\frac{k}{m}}u_0\right)t + u_0 \\ t &= -\frac{u_0}{v_0 + \sqrt{\frac{k}{m}}u_0} \end{aligned}$$

Therefore, the mass passes through the equilibrium position just once, provided that $u_0$ and $v_0$ are chosen so that this expression for $t$ is positive. If it is not positive, then the mass never passes through the equilibrium position.

**Case 2: Overdamping**

Suppose secondly that the motion is overdamped. Then the ratio of $\gamma^2$ to $4km$ is greater than 1, and $\gamma^2 - 4km > 0$.

$$mu'' + \gamma u' + ku = 0 \tag{2}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $u = e^{rt}$.

$$u = e^{rt} \quad \rightarrow \quad u' = re^{rt} \quad \rightarrow \quad u'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for $r$.

$$m(r^2 e^{rt}) + \gamma(re^{rt}) + k(e^{rt}) = 0$$

Divide both sides by $e^{rt}$.

$$mr^2 + \gamma r + k = 0$$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

$$r = \left\{ \frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m}, \frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} \right\}$$

Two solutions to equation (2) are

$$u = \exp\left(\frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m}t\right) \quad \text{and} \quad u = \exp\left(\frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m}t\right).$$

By the principle of superposition, the general solution for $u$ is a linear combination of these two.

$$u(t) = C_3 \exp\left(\frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m}t\right) + C_4 \exp\left(\frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m}t\right)$$

Differentiate it with respect to $t$.

$$u'(t) = C_3 \frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m} \exp\left(\frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m}t\right) + C_4 \frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} \exp\left(\frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m}t\right)$$

Now apply the initial conditions, $u(0) = u_0$ and $u'(0) = v_0$, to determine $C_3$ and $C_4$.

$$u(0) = C_3 + C_4 = u_0$$

$$u'(0) = C_3 \frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m} + C_4 \frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} = v_0$$

Solving this system of equations yields

$$C_3 = \frac{u_0\left(\sqrt{\gamma^2 - 4km} - \gamma\right) - 2mv_0}{2\sqrt{\gamma^2 - 4km}} \quad \text{and} \quad C_4 = \frac{u_0\left(\sqrt{\gamma^2 - 4km} + \gamma\right) + 2mv_0}{2\sqrt{\gamma^2 - 4km}}.$$

Thus, the amplitude of the overdamped system is

$$u(t) = \frac{u_0\left(\sqrt{\gamma^2 - 4km} - \gamma\right) - 2mv_0}{2\sqrt{\gamma^2 - 4km}} \exp\left(\frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m} t\right) + \frac{u_0\left(\sqrt{\gamma^2 - 4km} + \gamma\right) + 2mv_0}{2\sqrt{\gamma^2 - 4km}} \exp\left(\frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} t\right).$$

Set $u(t) = 0$ and solve for $t$ to determine the time at which the mass passes through the equilibrium position.

$$0 = \frac{u_0\left(\sqrt{\gamma^2 - 4km} - \gamma\right) - 2mv_0}{2\sqrt{\gamma^2 - 4km}} \exp\left(\frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m} t\right) + \frac{u_0\left(\sqrt{\gamma^2 - 4km} + \gamma\right) + 2mv_0}{2\sqrt{\gamma^2 - 4km}} \exp\left(\frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} t\right)$$

$$-\frac{u_0\left(\sqrt{\gamma^2 - 4km} - \gamma\right) - 2mv_0}{2\sqrt{\gamma^2 - 4km}} \exp\left(\frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m} t\right) = \frac{u_0\left(\sqrt{\gamma^2 - 4km} + \gamma\right) + 2mv_0}{2\sqrt{\gamma^2 - 4km}} \exp\left(\frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} t\right)$$

$$-\frac{\frac{u_0\left(\sqrt{\gamma^2 - 4km} - \gamma\right) - 2mv_0}{2\sqrt{\gamma^2 - 4km}}}{\frac{u_0\left(\sqrt{\gamma^2 - 4km} + \gamma\right) + 2mv_0}{2\sqrt{\gamma^2 - 4km}}} = \exp\left(\frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} t + \frac{\gamma + \sqrt{\gamma^2 - 4mk}}{2m} t\right)$$

$$\frac{2mv_0 - u_0\left(\sqrt{\gamma^2 - 4km} - \gamma\right)}{2mv_0 + u_0\left(\sqrt{\gamma^2 - 4km} + \gamma\right)} = \exp\left(\frac{\sqrt{\gamma^2 - 4mk}}{m} t\right)$$

$$\ln\left[\frac{2mv_0 - u_0\left(\sqrt{\gamma^2 - 4km} - \gamma\right)}{2mv_0 + u_0\left(\sqrt{\gamma^2 - 4km} + \gamma\right)}\right] = \frac{\sqrt{\gamma^2 - 4mk}}{m} t$$

$$t = \frac{m}{\sqrt{\gamma^2 - 4mk}} \ln\left[\frac{2mv_0 - u_0\left(\sqrt{\gamma^2 - 4km} - \gamma\right)}{2mv_0 + u_0\left(\sqrt{\gamma^2 - 4km} + \gamma\right)}\right]$$

Therefore, the mass passes through the equilibrium position just once, provided that $u_0$ and $v_0$ are chosen so that this expression for $t$ is real and positive. If it is not real and positive, then the mass never passes through the equilibrium position.